Journal of Statistical Physics, Vol. 81, Nos. 3/4, 1995

Correlations in Two-Component Log-Gas Systems

A. Alastuey[1] and P. J. Forrester[2]

Received January 18, 1995

A systematic study of the properties of particle and charge correlation functions in the two-dimensional Coulomb gas confined to a one-dimensional domain is undertaken. Two versions of this system are considered: one in which the positive and negative charges are constrained to alternate in sign along the line, and the other where there is no charge ordering constraint. Both systems undergo a zero-density Kosterlitz–Thouless-type transition as the dimensionless coupling $\Gamma := q^2/kT$ is varied through $\Gamma = 2$. In the charge-ordered system we use a perturbation technique to establish an $O(1/r^4)$ decay of the two-body correlations in the high-temperature limit. For $\Gamma \to 2^+$, the low-fugacity expansion of the asymptotic charge–charge correlation can be resummed to all orders in the fugacity. The resummation leads to the Kosterlitz renormalization equations. In the system without charge ordering the two-body correlations exhibit an $O(1/r^2)$ decay in the high-temperature limit, with a universal amplitude for the charge–charge correlation which is associated with the state being conductive. Low-fugacity expansions establish an $O(1/r^{\Gamma})$ decay of the two-body correlations for $2 < \Gamma < 4$ and an $O(1/r^4)$ decay for $\Gamma > 4$. For both systems we derive sum rules which relate the long-wavelength behaviour of the Fourier transform of the charge correlations to the dipole carried by the screening cloud surrounding two opposite internal charges. These sum rules are checked for specific solvable models. Our predictions for the Kosterlitz–Thouless transition and the large-distance behavior of the correlations should be valid at low densities. At higher densities, both systems might undergo a first-order liquid–gas transition analogous to the two-dimensional case.

KEY WORDS: Kosterlitz–Thouless transition; log-gas systems; correlations; fugacity expansions; sum rules.

[1] École Normale Supérieure de Lyon, Laboratoire de Physique, Unité de Recherche Associée 1325 au Centre National de la Recherche Scientifique, 69364 Lyon Cedex 07, France. E-mail: alastuey@physique.ens-lyon.fr.

[2] Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia. E-mail: matpjf@maths.mu.oz.au.

0022-4715/95/1100-0579$07.50/0

1. INTRODUCTION

The natural domain for a system of charges interacting via the two-dimensional (logarithmic) Coulomb potential is a plane. Notwithstanding this fact, there is still interest in studying the statistical mechanics of log-potential Coulomb systems confined to a line (we will refer to such systems as long-gases). Two-component log-gases have attracted interest because of some equivalences with models in solid-state physics (Kondo problem[1] and quantum Brownian motion problem[2]), and the fact that for low density a Kosterlitz–Thouless-type pairing transition takes place as the temperature is lowered below the critical coupling $\Gamma = 2$.[1,3]

In this paper we will study properties of correlations in the two-component log-gas with oppositely signed charges of strength q, and a variant of this system is which the positive and negative charges are confined to alternate in position along the line. For couplings $\Gamma := q^2/kT \geqslant 1$ we also assume that the particles are at the center of hard rods of length σ, which prevents short-distance collapse. So as to put our work in context, let us briefly review known properties of the critical behavior and correlations in two-component log-gas systems, and contrast this what is known about the two-component, two-dimensional Coulomb gas (2dCG) where appropriate.

1.1. The Charge-Ordered System

The system with charge ordering was first studied in the context of its application to the Kondo problem.[1] In this seminal work, Anderson *et al.* transformed the grand partition function of the system with a hard-rod length $\sigma + d\sigma$, fugacity ζ, and coupling Γ into the grand partition function of the system with hard-rod length σ and modified ζ and Γ, thereby deriving a pair of coupled renormalization equations. The transformation, which is approximate, requires ζ and $\Gamma - 2$ to be small and is thus applicable in the neighborhood of the Kosterlitz–Thouless transition. Remarkably, applying this procedure to study the 2dCG, Kosterlitz[4] found essentially the same equations. Furthermore, for the 2dCG it is well known (see, e.g., ref. 5) that a renormalization procedure can be applied to study the charge–charge correlation and a certain length-dependent dielectric constant. Again, by an appropriate choice of variables, the resulting equations are precisely those found by Anderson *et al.*

One immediate prediction from the renormalization equations is the dependence of the critical coupling (Γ_c) on fugacity: thus for the charge-ordered system

$$\Gamma_c - 2 = 2^{5/2}\zeta \tag{1.1}$$

valid to leading order in $\Gamma_c - 2$ and ζ. This phase boundary separates a low-temperature dipole phase in which the positive and negative charges are paired from a high-temperature phase in which the positive and negative charges are dissociated. More quantitatively, from the mapping to the Kondo problem, Schotte and Schotte[6] have argued that for the finite charge-ordered system of length L with periodic boundary conditions

$$\left\langle \left(\sum q_i x_i - L/2 \right)^2 \right\rangle \underset{L \to \infty}{\sim} \begin{cases} c(\Gamma) L & \text{for} \quad \Gamma < 2 \\ c(\Gamma) L^2 & \text{for} \quad \Gamma > 2 \end{cases} \tag{1.2}$$

This behavior, in which the boundary conditions play an essential role, was confirmed by Monte Carlo simulation.[6]

In contrast to the phase indicator (1.2), the phases of the 2dCG can be distinguished by the dielectric constant ε, which is finite in the dipole phase and infinite in the high-temperature (conductive) phase. We recall that if the system has dielectric constant ε, then a fraction $1 - 1/\varepsilon$ of an infinitesimal external charge density will be screened. Furthermore, ε can be expressed in terms of the particle correlations by the formulas[7]

$$\frac{1}{\varepsilon} = 1 + \frac{\pi\beta}{2} \int d\mathbf{r}\, r^2 C(r) \tag{1.3a}$$

and

$$\frac{1}{\varepsilon} = -\frac{\beta q}{2\rho} \int d\mathbf{r}\, \mathbf{F}(\mathbf{r})\, \mathbf{p}_{-+}(\mathbf{r})\, \rho_{-+}(r) \tag{1.3b}$$

where $C(r)$ denotes the charge–charge correlation, $\mathbf{F}(\mathbf{r})$ denotes the force corresponding to the two-body potential between two positive charges of unit strength (which is assumed to be smoothly regularized at the origin), $\rho_{-+}(r)$ denotes the distribution function between two opposite charges, and $\mathbf{p}_{-+}(\mathbf{r})$ denotes the dipole moment of the charge distribution into by two fixed opposite charges separated a distance r.

In log-gas systems it can be argued that a dielectric phase has $\varepsilon = 1$ (see, e.g., ref. 8). A linear response argument then gives

$$C(r) \sim o\left(\frac{1}{r^2}\right) \qquad \text{as} \quad r \to \infty \tag{1.4a}$$

On the other hand, in a conductive phase the charge–charge correlation must exhibit the universal asymptotic behavior[9]

$$C(r) \sim -\frac{q^2}{\Gamma(\pi r)^2} \tag{1.4b}$$

In the sense of screening an infinitesimal external charge, the charge-ordered system is expected to always exhibit a dielectric phase (except possibly at the phase boundary).[8] Hence, from (1.4a), the decay of $C(r)$ should be faster than $1/r^2$ in both the low-temperature dipole and high-temperature phase. When the positions of the particles are restricted to the sites of a lattice (this mimics a hard-core regularization), the isotherm $\Gamma = 1$ is exactly solvable.[9] Both the truncated two-particle distribution functions between like and opposite charges, and thus $C(r)$, exhibit an asymptotic $O(1/r^4)$ decay in accordance with this prediction.

1.2. No Charge Ordering

Application of the rescaling method of Anderson *et al.* to the two-component log-gas without charge ordering leads to the conclusion[3] that the critical coupling is $\Gamma = 2$ independent of the fugacity, in contrast to the behavior (1.1). No other information is obtained.

The charge–charge correlation function in the low-temperature dipole phase must again exhibit the behavior (1.4a). This has been explicitly verified for the solvable isotherm $\Gamma = 4$, where $O(1/r^4)$ decay was found.[10] The high-temperature phase is expected to be conductive and thus $C(r)$ should obey the sum rule (1.4b). The phase boundary $\Gamma = 2$, when the positions of the particles are restricted to the sites of a lattice, is a solvable isotherm[10, 11] and it is found that $C(r)$ has an $O(1/r^2)$ decay, but with a density dependent amplitude.

1.3. Outline of This Work

From the above brief review, it is clear that there are many gaps in our knowledge of the critical properties and the behavior of correlations in two-component log-gas systems. To improve on this situation, we will make a fairly systematic study of the correlation functions in the high- and low-temperature phases as well as in the vicinity of the zero-density critical point. The charge-ordered system is considered in Section 2, while the system without charge ordering is considered in Section 3. Concluding remarks are made in Section 4.

In Section 2.1 the two-particle correlations in the high-temperature phase of the charge-ordered system are analysed using a perturbative approach. The correlations in the scaling region are analyzed in Section 2.2 by applying the low-density resummation method of Alastuey and Cornu[7] and in Section 2.3 the correlations in the low-temperature phase away from criticality are considered. In Section 2.4 the second BGY equation is transformed into Fourier space and a sum rule analogous to (1.3b) is obtained.

In Section 3.1, the two-particle correlations in the conductive phase of the system without charge ordering are studied using a linear response argument and macroscopic electrostatics, and a high-temperature resummation technique. The asymptotic behavior of the correlations in the dielectric phase are determined in Section 3.2 by studying the low-fugacity expansions at $O(\zeta^4)$, and an analysis of the second BGY equation similar to that given in Section 2.3 for the charge-ordered model is made in Section 3.3. From the latter analysis the behavior of the three- and four-particle correlations in the dielectric phase is deduced.

In Section 4 we summarize our results with emphasis on the mechanisms behind the contrasting behaviors of the correlations in the log-gas with and without charge ordering.

2. THE CHARGE-ORDERED LOG-GAS

2.1. Decay of Correlations in the High-Temperature Phase

Along the high-temperature solvable isotherm $\Gamma = 1$ it has shown by explicit calculation[9] that the charge–charge correlation decays as $O(1/r^4)$, and thus by (1.4a) the system does not screen an infinitesimal external charge. Indeed, because of the charge ordering constraint, it was argued[9] that this latter property is a general feature of the system for all temperatures. In particular, the high-temperature phase is not conductive, which suggests that the conventional methods of analyzing the high-temperature phase in Coulomb systems (Debye–Hückel-type theories, etc.) are not applicable. In fact the familiar Abe–Meeron[12] diagrammatics, with Debye-like mean-field potentials resulting from chain resummations, cannot be applied here because the constraint of charge ordering is equivalent to introducing a many-body potential between the charges, whereas the diagrammatics holds for systems with two-body forces only. We use instead a perturbative method which relies on the special screening properties of the $\Gamma = 0$, contrained two-species perfect-gas reference system.

2.1.1. Correlations at $\Gamma = 0$. For $\Gamma < 1$ in general, and $\Gamma = 0$ in particular, there is no need to regularize the short-range singularity of the logarithmic potentials, as the corresponding Boltzmann factors are integrable. Thus at $\Gamma = 0$ the log-gas system can be considered as a two-species perfect gas of point particles, constrained so that the two species alternate in position along the line. Let us suppose the first particle belongs to species $+$ while the second particle belongs to species $-$ (these boundary conditions will result in a symmetry breaking: the ρ_{+-} and ρ_{-+} correlation functions will not in general be equal), and there are N particles of each species with coordinates $x_1,..., x_N$ (+ species) and $y_1,..., y_N$ (− species), where $x_j, y_j \in [-L/2, L/2]$ $(j=1,..., N)$. The canonical partition function Z_{2N} for this system is defined as

$$Z_{2N} := \left(\frac{1}{N!}\right)^2 \int^* dx_1 \cdots dy_N \tag{2.1a}$$

where * denotes the interlacing constraint on the interval $[-L/2, L/2]$. It is easily evaluated to give

$$Z_{2N} = \frac{1}{(2N)!} L^{2N} \tag{2.1b}$$

The two-particle distribution $\rho_{++}(x_1, x_2)$ is given by

$$\rho_{++}(x_1, x_2) := \frac{N(N-1)}{Q_{2N}} \int^{*(x_1, x_2)} dx_3 \cdots dy_N \tag{2.2a}$$

where

$$Q_{2N} = (N!)^2 Z_{2N} \tag{2.2b}$$

and $*(x_1, x_2)$ denotes the interlacing constraint, given that there are particles of species + at x_1 and x_2. The distribution $\rho_{-+}(x_1, y_1)$ is defined similarly.

To calculate $\rho_{++}(x_1, x_2)$, we note that in the interval $[-L/2, x_1]$ there must be equal numbers of particles of each species, while in the intervals $[x_1, x_2]$ and $[x_2, L/2]$ there must be one more particle of species − than particle of species +. Let the number of particles of species + in each interval be M_a, M_b, and M_c, respectively (note that $M_a + M_b + M_c = N-2$). Then we have

$$\rho_{++}(x_1, x_2) = \frac{(N!)^2}{Z_{2N}} \sum_{\substack{M_a, M_b, M_c = 0 \\ M_a + M_b + M_c = N-2}}^{N-2} \frac{(x_1 + L/2)^{2M_a} (x_2 - x_1)^{2M_b + 1} (L/2 - x_2)^{2M_c + 1}}{(2M_a)! (2M_b + 1)! (2M_c + 1)!} \tag{2.3}$$

Substituting (2.1b), we can write this as

$$\rho_{++}(x_1, x_2) = \frac{2N(N-1)}{L^{2N}} S_{2N-2}\left(x_1 + \frac{L}{2}, x_2 - x_1, \frac{L}{2} - x_2\right) \tag{2.4a}$$

where

$$S_p(a, b, c)$$

$$:= \sum_{\substack{M_a, M_b, M_c \\ M_a + M_b + M_c = p/2 - 1}}^{p/2-1} \frac{p!}{(2M_a)!(2M_b+1)!(2M_c+1)!} a^{2M_a} b^{2M_b+1} c^{2M_c+1} \tag{2.4b}$$

Using the generalized binomial expansion

$$(a+b+c)^p = \sum_{\substack{p_1, p_2, p_3 = 0 \\ p_1 + p_2 + p_3 = 0}}^{p} \frac{p!}{p_1!\, p_2!\, p_3!} a^{p_1} b^{p_2} c^{p_3} \tag{2.5}$$

it is straightfoward to derive the summation formula

$$S_p(a, b, c) = \tfrac{1}{4}[(a+b+c)^p + (-a+b+c)^p - (a-b+c)^p - (-a-b+c)^p] \tag{2.6}$$

valid for p even. With this result the thermodynamic limit in (2.4a) can be taken immediately to give

$$\rho_{++}(0, x) = \rho^2[1 - e^{-4\rho |x|}] \tag{2.7}$$

A similar calculation shows

$$\rho_{+-}(0, x) = \rho^2[1 + e^{-4\rho |x|}] \tag{2.8}$$

or alternatively this result could be deduced from the requirement

$$\tfrac{1}{2}[\rho_{++}(0, x) + \rho_{+-}(0, x)] = \rho^2 \tag{2.9}$$

which follows since the combination of two-particle distributions on the l.h.s. gives the two-particle distribution of the (unconstrained) perfect gas.

The crucial feature of these distributions is that they exhibit perfect screening of an internal "charge." Thus

$$\int_{-\infty}^{\infty} dx\, [\rho_{++}(0, x) - \rho_{+-}(0, x)] = -\rho \tag{2.10}$$

This property allows the correlations in the high-temperature limit to be studied by a perturbation expansion about the $\Gamma = 0$ constrained perfect-gas reference system.

2.1.2. Perturbation About $\Gamma = 0$. In any two-component log-gas system, the Coulomb interaction energy U can be written

$$U = \frac{1}{2} \int dx \int dx' \, [Q(x)\, Q(x')]_{\text{nc}} \, v_c(|x - x'|) \tag{2.11a}$$

where

$$v_c(|x - x'|) := -\log |x - x'| \tag{2.11b}$$

$$Q(x) := q \sum_{j=1}^{N} [\delta(x - x_j) - \delta(x - y_j)] \tag{2.11c}$$

with the positive (negative) species at $x_j (y_j)$ and

$$[Q(x)\, Q(x')]_{\text{nc}}$$

denotes that the product $Q(x)\, Q(x')$ is formed with products over coincident points excluded. In the charge-ordered two-component system the truncated two-particle distribution between like charges is given by

$$\rho^T_{++}(0, x_a) = \frac{\langle [N_+(0)\, N_+(x_a)]_{\text{nc}} e^{-\beta U} \rangle_0}{\langle e^{-\beta U} \rangle_0} - \frac{\langle N_+(0)\, e^{-\beta U} \rangle_0 \langle N_+(x_a)\, e^{-\beta U} \rangle_0}{\langle e^{-\beta U} \rangle_0^2} \tag{2.12}$$

where the subscript 0 indicates that the averages are taken with respect to the constrained perfect-gas reference system and

$$N_+(x_a) = \sum_{j=1}^{N} \delta(x - x_j)$$

Next we expand the exponential in (2.12) to leading order in β. We obtain

$$\rho^T_{++}(0, x_a) \sim -\frac{\beta}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dx' \times \langle N_+(0)\, N_+(x_a) [Q(x)\, Q(x')]_{\text{nc}} \rangle_0^T \, v_c(|x - x'|) \tag{2.13}$$

where the truncation is defined with respect to the three quantities 0, x_a, and (x, x').[13] When expressed in terms of the fully truncated Ursell functions there are two contributions. One involves only two-body Ursell functions, while the other involves the four-body Ursell function. Since the Ursell functions in the reference system decay exponentially, the leading contribution comes from the term involving the two-body functions. Hence (2.13) can be rewritten as

$$\rho^T_{++}(0, x_a) \sim -\beta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, dx' \\ \times \langle N_+(0)\, Q(x) \rangle_0 \langle N_+(x_a)\, Q(x') \rangle_0\, v_c(|x-x'|) \qquad (2.14)$$

The two-body averages are given by

$$\langle N_+(0)\, Q(x) \rangle_0 = q(\rho^{(0)}_{++}(x) - \rho^{(0)}_{-+}(x) + \rho_0\, \delta(x)) \\ = -2q\rho_0^2 e^{-4\rho_0 |x|} + q\rho_0\, \delta(x) \qquad (2.15\text{a})$$

and

$$\langle Q(x')\, N_+(x_a) \rangle_0 = \langle Q(x'-x_a)\, N_+(0) \rangle_0 \qquad (2.15\text{b})$$

To calculate the leading large-x_a behavior of (2.14), we make the expansion

$$v_c(|x-x'|) = v_c(|x_a|) + \sum_{n=1}^{\infty} \frac{(x'-x-x_a)^n}{n!} \frac{\partial^n}{\partial x^n} v_c(|x|) \bigg|_{x=x_a} \qquad (2.16)$$

Due to the perfect screening property (2.10) of the reference system and the fact the distribution functions in the reference system are even in their argument, we see that the first nonzero term which results from substituting (2.16) in (2.14) is $n=4$. Noting

$$\frac{\partial^4}{\partial x^4} v_c(|x|) = -\frac{\partial^4}{\partial x^4} \log |x| = \frac{3!}{x^4}$$

we therefore have that to first order in Γ and for large x_a

$$\rho^T_{++}(0, x_a) \sim -\frac{6\Gamma\rho_0^2}{(4\rho_0 x_a)^4} \qquad (2.17)$$

The leading asymptotics of $\rho^T_{+-}(0, x_a)$ can be computed in a similar way. In fact, since in the reference system

$$\rho^T_{+-}(0, x) = -\rho^T_{++}(0, x)$$

the only difference in the calculation is a minus sign, so we obtain

$$\rho^T_{+-}(0, x) \sim -\rho^T_{++}(0, x) \sim \frac{6\Gamma\rho_0^2}{(4\rho_0 x_a)^4} \qquad (2.18)$$

Although (2.17) and (2.18) are valid of first order in Γ only, we expect the leading $O(1/x^4)$ decay to persist in the high-temperature phase for $\Gamma \leqslant 1$ at least, as the exact result at $\Gamma = 1$[10] exhibits this behavior. Indeed, we expect all terms in the Γ-expansion of the truncated two-body distributions to decay as $1/x^4$; this would follow from a diagrammatic analysis similar to the one used by Alastuey and Martin.[13] The exact result also exhibits the property (2.18) relating the leading asymptotics of $\rho^T_{+-}(0, x)$ and $\rho^T_{++}(0, x)$, which suggests this may also be a general property of the high-temperature phase for $\Gamma \leqslant 1$ at least.

2.2. Correlations in the Scaling Region of the Low-Temperature Dipole Phase

For the 2dCG it has been proved[14] that all the coefficients in the low-fugacity expansion of the pressure and the correlation functions are convergent in the dipole phase ($\Gamma \geqslant 4$). We expect this property to remain true in the low-temperature dipole phase of the log-gas systems ($\Gamma \geqslant 2$). With this assumption, we can consider the asymptotic large-r behavior of each such coefficient in the charge–charge correlation $C(r)$ for $\Gamma \geqslant 2$. In particular we can consider the behavior for $\Gamma \to 2^+$ as the Kosterlitz–Thouless transition is approached from the dipole phase. Motivated by the analogy between critical features of the alternating model and the 2dCG (in particular the occurrence of the same renormalization equations), we do this by closely following the strategy of Alastuey and Cornu[7] in their analysis of $C(r)$ in the two-dimensional system.

For this purpose we introduce the asymptotic charge density $C_\Delta(r)$. This is defined as the terms in the asymptotic expansion of $C(r)$ which, when replacing $C(r')$ in

$$\Delta := 1 + \frac{4\Gamma}{q^2} \int_\sigma^\infty dr'\, r' C(r') \tag{2.19}$$

give the correct leading-order singular behavior of Δ for $\Gamma \to 2^+$ at each order in ζ. Note from (1.3a) that Δ is the analog of $1/\varepsilon$ in the 2dCG. Furthermore, in the low-fugacity limit, $\Delta - 1$ is proportional to the mean distance of separation between neighboring charges.[15] Our objective is to calculate $C_\Delta(r)$ and Δ in the scaling region of the low-temperature dipole phase.

2.2.1. Low-Fugacity Expansions. The low-fugacity expansions of the correlations can be obtained from the low-fugacity expansion of the logarithm of the grand partition function, with each positive (negative) charge given a position-dependent fugacity $\zeta a(x_j)$ ($\zeta b(y_j)$). The grand partition function is given by

$$\Xi[a, b] = \sum_{N=0}^{\infty} \zeta^{2N} Z_{2N}[a, b] \tag{2.20}$$

where $Z_{2N}[a, b]$ denotes the partition function for equal number N of positive and negative charges. The truncated distributions can be calculated from (2.20) according to the formula

$$\rho^T_{+\cdots+-\cdots-}(r_1,\dots, r_{n_1}; s_1,\dots, s_{n_2}) = \frac{\delta^{n_1+n_2}}{\delta a(r_1)\cdots\delta a(r_{n_1})\,\delta b(s_1)\cdots\delta b(s_{n_2})} \log \Xi[a, b]\Big|_{a=b=1} \tag{2.21}$$

Now from (2.20)

$$\log \Xi[a, b] = \zeta^2 Z_2[a, b] + \zeta^4 (Z_4[a, b] - \tfrac{1}{2}(Z_2[a, b])^2) + O(\zeta^6) \tag{2.22}$$

Using (2.22) in (2.21), assuming that the leftmost particle always has a positive charge, and writing the partition functions explicitly, we obtain for the two-particle correlations

$$\rho^T_{++}(r) = \zeta^4 \left[\int_\sigma^{r-\sigma} dy_1 \int_{r+\sigma}^{\infty} dy_2 \left(\frac{r(y_2 - y_1)}{(y_2 - r)\, y_2 (r - y_1)\, y_1} \right)^\Gamma - \left(\int_\sigma^\infty \frac{dy}{y^\Gamma} \right)^2 \right] + O(\zeta^6) \tag{2.23a}$$

$$\begin{aligned}\rho^T_{+-}(r) = \frac{\zeta^2}{r^\Gamma} + \zeta^4 \Bigg\{ &\int_{2\sigma}^{r-\sigma} dx \int_\sigma^{x-\sigma} dy \left(\frac{(r-y)x}{r(x-y)(r-x)\, y} \right)^\Gamma - \left(\int_\sigma^\infty \frac{dy}{y^\Gamma} \right)^2 \\ &+ 2 \int_{r+2\sigma}^{\infty} dy \int_{\sigma+r}^{y-\sigma} dx \left[\left| \frac{x(y-r)}{(y-x)\, y(x-r) r} \right|^\Gamma - \left(\frac{1}{(y-x) r} \right)^\Gamma \right] \\ &- \frac{1}{r^\Gamma} \left(\int_{-\sigma}^{r+2\sigma} dy \int_{-\infty}^{y-\sigma} dx + \int_{r+\sigma}^{\infty} dy \int_{-\infty}^{r} dx \right) \frac{1}{(y-x)^\Gamma} \Bigg\} \\ &+ O(\zeta^6)\end{aligned} \tag{2.23b}$$

$$\rho^T_{-+}(r) = \zeta^4 \left[\int_{r+\sigma}^{\infty} dy \int_{-\infty}^{-\sigma} dx \times \left| \frac{(r-x)\, y}{r(y-x)(y-r)\, x} \right|^\Gamma - \left(\int_\sigma^\infty \frac{dy}{y^\Gamma} \right)^2 \right] + O(\zeta^6) \tag{2.23c}$$

where it is assumed that $r > 2\sigma$ in (2.23a) and $r > \sigma$ in (2.23b) and (2.23c) (for r less than these values the respective full distribution functions vanish). Also, the first double integral in (2.23b) is to be omitted if $r < 2\sigma$. To the same order, the low-fugacity expansion of the charge–charge correlation follows immediately from the formula

$$C(r) = 2q^2\rho\delta(r) + q^2[2\rho^T_{++}(r) - \rho^T_{+-}(r) - \rho^T_{-+}(r)] \tag{2.24}$$

2.2.2. Evaluation of $C_\Delta^{(2)}(r)$ and $C_\Delta^{(4)}(r)$. Let us denote the term proportional to ζ^{2j} in the low-fugacity expansions of $C(r)$, $C_\Delta(r)$, and Δ by $C^{(2j)}(r)$, $C_\Delta^{(2j)}(r)$ and $\Delta^{(j)}$, respectively. From (2.24) and (2.23) we have, for $r > \sigma$,

$$C^{(2)}(r) = q^2 \frac{\zeta^2}{r^\Gamma} \tag{2.25}$$

Substituting in (2.19) gives

$$\Delta^{(2)} = 1 + \frac{4\Gamma\zeta^2\sigma^{2-\Gamma}}{\Gamma - 2} \tag{2.26}$$

We note that $\Delta^{(2)}$ is singular as $\Gamma \to 2^+$ and furthermore to leading order is independent of σ. Both features are true of $\Delta^{(2n)}$ in general. The latter feature implies that only the large-r portion of $C^{(2n)}(r)$ contributes to the leading-order singular behavior of $\Delta^{(2n)}$, and thus $C_\Delta^{(2n)}(r)$ consists of terms in the asymptotic expansion of $C^{(2n)}(r)$. With $n = 1$ there is only one term in the asymptotic expansion, which is $C^{(2)}(r)$ itself, so trivially $C_\Delta^{(2)}(r) = C^{(2)}(r)$.

The analysis of the large-r behavior of the integrals in (2.23), which from (2.24) form $C^{(4)}(r)$, is done in Appendix A. There it is deduced that

$$C_\Delta^{(4)}(r) = -q^2\zeta^4 \frac{4\Gamma}{r^{2\Gamma-2}} \left\{ -\frac{1}{(\Gamma-2)^2}\left[\left(\frac{\sigma}{r}\right)^{2-\Gamma} - 1\right] + \frac{1}{\Gamma-2}\left(\frac{\sigma}{r}\right)^{2-\Gamma} \log r \right\} \tag{2.27}$$

and the integral representation

$$C_\Delta^{(4)}(r) = -q^2\zeta^4 \frac{1}{r^\Gamma} \int_{r+\sigma}^{2r} dx \int_{x+\sigma}^{2x-r} dy\, \mathcal{S}_{(0,r)}(x, y) \tag{2.28a}$$

with

$$\mathscr{S}_{(0,r)}(x, y) := \frac{4\Gamma}{(x-r)(y-x)^{\Gamma-1}} \tag{2.28b}$$

is also derived. The r.h.s. of (2.27) has the remarkable property of having an identical structure to the corresponding expression in the low-fugacity expansion of the charge–charge correlation in the 2dCG [ref. 7, Eq. (4.13)]. Furthermore, the integral expression (2.28) can be interpreted as the contribution of a mobile positive–negative dipole (positive charge at x, negative charge at y) pair partially screening (note that in all phases the external charges are not screened at all since $\varepsilon = 1$) the root dipole of separation r (positive charge at 0, negative charge at r). The distance of separation $|y-x|$ between charges within the mobile dipole is constrained to be less than the closest distance between this dipole and the root charges at 0 or r, and this closest distance is to be no greater than r. The factor of 4 comes from the 4 equivalent ways of arranging the mobile dipole about the root charges (the screening dipole may lie close to 0 or r, and inside or outside the root dipole). This "nested" dipole interpretation of $C_\Delta^{(4)}(r)$ is analogous to that found in ref. 7 for the quantity $C_\varepsilon^{(4)}(r)$ in the 2dCG.

2.2.3. Nested Dipole Chain Hypothesis. Although we have calculated $C_\Delta^{(4)}(r)$, it may seen a formidable task to do likewise for $C_\Delta^{(2j)}(r)$, $j \geqslant 3$. Indeed, an analysis similar to Appendix A does not appear to be feasible. Instead, having identified an analogy between $C_\Delta^{(4)}(r)$ and $C_\varepsilon^{(4)}(r)$ of the 2dCG, we adapt the method given in ref. 7 to calculate $C_\varepsilon^{(2j)}(r)$.

The basis of the method of ref. 7 is the hypothesis, which has its origin in the work of Kosterlitz and Thouless,[16] that the configurations contributing to $C_\Delta^{(2j)}(r)$ are all nested chains of dipoles, with the fixed dipole (positive charge at 0 and negative charge at r) the largest. The screening operator (2.28b) acts between dipoles connected in a chain. Furthermore, by including the factor of 4 in (2.28b), these chains can all be ordered to the right of the fixed negative charge at r. For example, at $O(\zeta^6)$, there are two distinct chains as given in Fig. 1.

The contributions to $C^{(6)}(r)$ from these chains are

$$\int_{r+\sigma}^{2r} dx_2 \int_{x_2+\sigma}^{2x_2-r} dy_2\, \mathscr{S}_{(0,r)}(x_2, y_2) \int_{y_2+\sigma}^{2y_2-x_2} dx_1 \int_{x_1+\sigma}^{2x_1-y_2} dy_1\, \mathscr{S}_{(x_2,y_2)}(x_1, y_1)$$

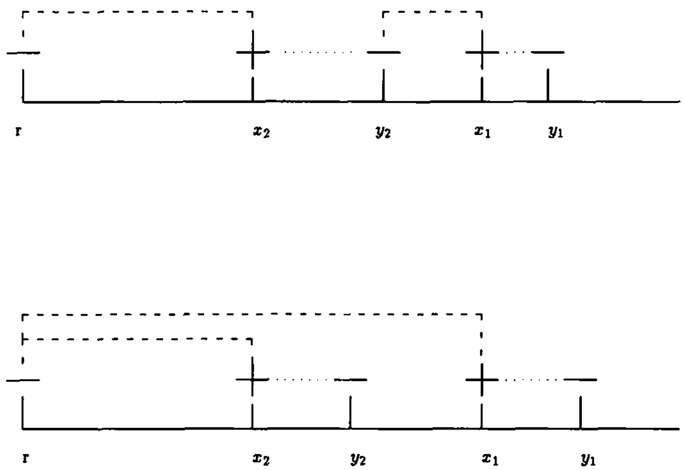

Fig. 1. The two distinct nested dipoles at $O(\zeta^6)$. The square braces indicate a screening operator which connects the charge on the left and with the dipole on the right end.

and

$$\left[\int_{r+\sigma}^{2r} dx_2 \int_{x_2+\sigma}^{2x_2-r} dy_2\, \mathscr{S}_{(0,r)}(x_2, y_2)\right]\left[\int_{r+\sigma}^{2r} dx_1 \int_{x_1+\sigma}^{2x_1-r} dy_1\, \mathscr{S}_{(0,r)}(x_1, y_1)\right]$$

Furthermore, the first of these contributions needs to be weighted by a factor of 2 to account for the relabeling degeneracy, and this linear combination needs to be multiplied by a factor of $-q^2\zeta^6/(4!\, r^\Gamma)$.

At general order, the nested chain hypothesis gives [ref. 7, Eqs. (4.26)–(4.28)]

$$C_\Delta^{(2n)}(r) = -\frac{q^2\zeta^2}{r^\Gamma}\frac{\zeta^{(2n-2)}}{(2n-2)!} S_\Delta^{(2n-2)}(r) \tag{2.29a}$$

where

$$S_\Delta^{(2n-2)}(r) = \sum_{p=1}^{n-1} \frac{(n-1)!}{p!\,(n-1-p)!} \times \sum_{\substack{q_\alpha \geqslant 0 \\ q_1+\cdots+q_p=n-1-p}} \frac{(n-1-p)!}{q_1!\cdots q_p!} I_{2q_1}(r)\cdots I_{2q_p}(r) \tag{2.29b}$$

with

$$I_{2q}(r) = \int_{r+\sigma}^{2r} dx \int_{x+\sigma}^{2x-r} dy\, \mathscr{S}_{(0,r)}(x, y)\, S_{\Delta}^{(2q)}(x-y) \tag{2.29c}$$

From the structure (2.29) it is shown in ref. 7 that the series

$$C_{\Delta}(r) = \sum_{n=1}^{\infty} C_{\Delta}^{(2n)}(r) \tag{2.30}$$

can be summed whatever the form of $\mathscr{S}_{(0,r)}(x, y)$, with the result

$$C_{\Delta}(r) = -\frac{q^2\zeta^2}{r^{\Gamma}} \exp\left[-\frac{1}{q^2} \int_{r+\sigma}^{2r} dx \int_{x+\sigma}^{2x-r} dy \right.$$
$$\left. \times \mathscr{S}_{(0,r)}(x, y)(y-x)^{\Gamma}\, C_{\Delta}(y-x) \right] \tag{2.31}$$

Since $\mathscr{S}_{(0,r)}(x, y)$ is explicitly given by (2.27b), we see that it is possible to simplify the double integral in (2.31) by an integration by parts in the y variable. This gives

$$C_{\Delta}(r) = -\frac{q^2\zeta^2}{r^{\Gamma}} \exp\left[-4\beta \log r \int_{\sigma}^{r} dx\, xC_{\Delta}(x) + 4\beta \int_{\sigma}^{r} dx \log x\, xC_{\Delta}(x) \right] \tag{2.32}$$

This expression has the same structure as that of $C_{\varepsilon}(r)$ for the 2dCG [ref. 7, Eq. (4.32)].

Introducing the length-dependent version of (2.19) by

$$\Delta(r) := 1 + \frac{4\Gamma}{q^2} \int_{\sigma}^{r} dr'\, r' C_{\Delta}(r') \tag{2.33}$$

which gives the contribution to Δ from all particles of separation at most r and is analogous to the length-dependent dielectric constant defined by Kosterlitz and Thouless,[16] we can obtain from (2.32) and (2.33) the coupled differential equations

$$\frac{d}{d\log(r)} \Delta(r) = \frac{4\Gamma}{q^2} r^2 C_{\Delta}(r) \tag{2.34a}$$

and

$$\frac{d}{d\log(r)} C_{\Delta}(r) = -\Gamma C_{\Delta}(r)\, \Delta(r) \tag{2.34b}$$

Making the change of variables

$$y(r)=\frac{2}{q}[-\Gamma C_{\Delta}(r)]^{1/2}r, \qquad t(r)=1-\frac{\Gamma}{2}\Delta(r), \qquad l=\log r$$

we find that the differential equations (2.34) read

$$\frac{d}{dl}y(l)=y(l)\,t(l) \tag{2.35a}$$

$$\frac{d}{dl}t(l)=[y(l)]^2 \tag{2.35b}$$

These equations, with different meanings for $y(l)$ and $t(l)$, are precisely those obtained by Anderson *et al.*[1]

It is simple to solve the system (2.35) for $t(l)$ in terms of $y(l)$, where the constant of integration can be determined by putting $r=\sigma$ in the integral equation. We find, using r instead of l,

$$-\frac{4\Gamma}{q^2}r^2C_{\Delta}(r)=\left[1-\frac{\Gamma}{2}\Delta(r)\right]^2+4\Gamma\zeta^2\sigma^{2-\Gamma}-\left(1-\frac{\Gamma}{2}\right)^2 \tag{2.36}$$

Now, assuming $\Gamma>2$, we have from the low-fugacity expansion that $C_{\Delta}(r)$ is $o(1/r^2)$ as $r\to\infty$. Recalling the definition (2.19) and taking the limit $r\to\infty$ in (2.36), we thus have

$$\Delta=1-\frac{\Gamma-2}{\Gamma}\left\{\left[1-\frac{\Gamma(4\zeta)^2\sigma^{2-\Gamma}}{(\Gamma-2)^2}\right]^{1/2}-1\right\} \tag{2.37}$$

Notice that the radius of convergence of the resummed function in (2.37) gives the phase boundary (1.1). This expression for Δ has an identical structure to that of $1/\varepsilon$ in the scaling region of the dipole phase of the 2dCG.[7]

The resummation of $C_{\Delta}(r)$ can be performed by analyzing the differential equations (2.34). This has been done in ref. 7 using these equations as they apply to the correlations of the 2dCG. From the working of ref. 7 [Eqs. (4.41)–(4.50)] we find

$$C_{\Delta}(r)=-q^2\zeta^2\left[A_0\left(\frac{\sigma}{r}\right)^{\Gamma\Delta}+\sum_{N=1}^{\infty}A_N\left(\frac{\sigma}{r}\right)^{\Gamma\Delta+N(\Gamma\Delta-2)}\right] \tag{2.38a}$$

where

$$A_0=\exp\left\{-\Gamma\int_{\sigma}^{\infty}\frac{dt}{t}[\Delta(t)-\Delta]\right\} \tag{2.38b}$$

and the A_N can be expressed in terms of A_0 and the function

$$\frac{2(4\zeta)^2/(\Gamma-2)^2}{[1-2(4\zeta)^2/(\Gamma-2)^2]^{1/2}} \tag{2.39}$$

Thus the coupling Γ is renormalized as the density-dependent quantity $\Gamma\Delta$, and $\Gamma\Delta$ is the power involved in the expansion (2.38) of $C_\Delta(r)$.

2.3. Correlations in the Low-Temperature Dipole Phase Away from Criticality

The analysis of Appendix A identifies the leading large-r expansion of $\rho^T_{++}(r)$, $\rho^T_{+-}(r)$, and $\rho^T_{-+}(r)$ at $O(\zeta^4)$ throughout the low-temperature dipole phase $\Gamma > 2$. Thus from the expansions (A6) we have

$$\rho^{T(4)}_{++}(r) \sim \rho^{T(4)}_{+-}(r) \sim \rho^{T(4)}_{-+}(r) \sim \frac{\zeta^4 \Gamma}{(\Gamma-2)^2 r^2} \tag{2.40}$$

In the case of $\rho^{T(4)}_{++}(r)$ this behavior results from the contribution of the configuration A of Fig. 2 in Appendix A. Similarly, configurations in which each root particle is paired with a mobile particle to form a neutral cluster give the leading contribution to $\rho^{T(4)}_{+-}(r)$ and $\rho^{T(4)}_{-+}(r)$. These configurations have the same statistical weight at large distances, which implies the same behavior (2.40) for $\rho^{T(4)}_{++}$, $\rho^{T(4)}_{+-}$, and $\rho^{T(4)}_{-+}$.

To calculate the large-r behavior of $\rho^{T(2n)}_{++}(r)$, etc., for $n > 2$ we need to hypothesize the generalization of configuration A which is dominant in the respective integral formulas for $r \to \infty$. This generalization is to have all particles belonging to a neutral cluster about and including one or another of the root particles, with the interparticle spacing within a cluster small in comparison to r. The potential V_{2N} between clusters can be written as

$$V_{2N} = W_0 + W_r + U_{0r} \tag{2.41}$$

where W_0 and W_r denote the electrostatic energies of the individual clusters about and including the root particles at 0 and r, respectively, and U_{0r} denotes the mutual interaction. For large r, U_{0r} is to leading order a dipole–dipole potential:

$$U_{0r} \sim p_0 \frac{\partial}{\partial x_0} p_r \frac{\partial}{\partial r} \log |x_0 - r| \Big|_{x_0=0} = -\frac{p_0 p_r}{r^2} \tag{2.42}$$

where p_0 and p_r are the dipoles of the clusters about and including the roots particles. The situation for $n > 2$ is thus completely analogous to the case $n=2$ [recall the text below (A2)]. Insertion in the corresponding integral formulas and use of the expansion

$$e^{-\beta U_{0r}} \sim 1-\beta U_{0r}+\cdots \tag{2.43}$$

thus leads to the results

$$\rho_{++}^{T(2n)}(r) \sim \rho_{+-}^{T(2n)}(r) \sim \rho_{-+}^{T(2n)}(r) \sim \frac{\zeta^{2n}\alpha_{2n}(\Gamma)}{r^2} \tag{2.44}$$

where the coefficient $\alpha_{2n}(\Gamma)$ diverges as $\Gamma \rightarrow 2$. This $O(1/r^2)$ behavior can be understood as resulting from the dipole–dipole interaction between the neutral clusters at large separation.

The configurations giving the leading large-r behavior of higher order correlations will again be neutral clusters and, analogous to the case $n=2$, the potential between far-away neutral clusters will again be, to leading order, due to the dipole–dipole interaction and thus $O(1/r^2)$. Hence we expect

$$\rho_{+\cdots+-\cdots-}^{T(2n)}(x_1,\ldots,x_{n_1};y_1,\ldots,y_{n_2}) \sim \frac{\zeta^{2n}}{\prod x_{jk}^2 \prod y_{j'k'}^2 \prod |x_j-y_{k'}|^2} \tag{2.45}$$

as $x_{jk}:=x_k-x_j$, $y_{j'k'}:=y_{k'}-y_{j'}$, $|x_j-y_{k'}| \rightarrow \infty$, where we have omitted the amplitude. This asymptotic behavior requires at least one mobile particle about each root particle, and thus does not necessarily apply for $n<n_1+n_2$.

2.4. A Sum Rule from the BGY Equation for $C(x)$

It has been shown in ref. 7 that the BGY equation for the charge–charge correlation $C(r)$ in the 2dCG can be used to derive a sum rule which expresses the dielectric constant in terms of the dipole moment $p_{+-}(r)$ of the screening cloud surrounding two internal charges of opposite sign. In this subsection we will apply an analogous analysis to $C(x)$ (we use x to denote any position coordinate, positive or negative, and r to denote a nonnegative quantity) for the charge-ordered system, which leads to a sum rule involving the distribution functions between charges of opposite sign and the dipole moments $p_{+-}(x)$ and $p_{-+}(x)$.

The BGY equation for $C(r)$ in the 2dCG has been derived from the BGY equations for the two-particle distributions. The BGY equation for $C(x)$ in the two-component log-gas without charge ordering can immediately be read off from the 2dCG result [ref. 7, Eq. (5.10)]. However, the resulting equation is not applicable to the charge-ordered system for two reasons: (i) the charge-ordered constraint introduces extra terms in the BGY equation for the two-particle distributions and thus for $C(x)$, and (ii) the symmetry $\rho_{+-}(x)=\rho_{-+}(x)$ assumed in the derivation is not valid in the low-temperature phase of the charge-ordered system.

To derive the BGY equations for $C(x)$ in the charge-ordered system let us then reconsider the BGY equations for the two-particle distribution functions. We suppose the logarithmic potential is regularized by hard cores of length σ symmetrically placed about each particle. We know that without charge ordering and with smoothly regularized potentials

$$v_{s_1 s_2}(x)=s_1 s_2 v(x)$$

the BGY equation for the distribution $\rho^T_{s_1 s_2}(x)$ can be written in the form [ref. 7, Eqs. (5.8a), (5.8b)]

$$\begin{aligned}
&\frac{\partial}{\partial x_2}\rho^T_{s_1 s_2}(x_{12})\\
&=s_1 s_2 \Gamma F(x_{12})\,\rho^T_{s_1 s_2}(x_{12})+s_2\beta q\rho^2\int_{-\infty}^{\infty} dx_3\,F(x_{32})\,Q_{s_1}(x_1 \mid x_3)\\
&\quad+s_2\Gamma\int_{-\infty}^{\infty} dx_3\,F(x_{32})[\rho^T_{s_1 s_2 +}(x_1, x_2, x_3)-\rho^T_{s_1 s_2 -}(x_1, x_2, x_3)] \qquad (2.46a)
\end{aligned}$$

where

$$F(x):=-\frac{\partial}{\partial x}v(x), \qquad x_{ab}:=x_b-x_a \qquad (2.46b)$$

and $Q_{s_1\dots s_n}(x_1,\dots, x_n \mid x)$ denotes the total charge density induced at x given that there are charges $s_1,\dots, s_n$ fixed at $x_1,\dots, x_n$:

$$\begin{aligned}
Q_{s_1\dots s_n}(x_1,\dots, x_n \mid x)&=q\,\frac{[\rho_{s_1\dots s_n +}(x_1,\dots, x_n, x)-\rho_{s_1\dots s_n -}(x_1,\dots, x_n, x)]}{\rho_{s_1\dots s_n}(x_1,\dots, x_n)}\\
&\quad+q\sum_{i=1}^{n} s_i\delta(x-x_i) \qquad (2.46c)
\end{aligned}$$

By considering the definition of $\rho_{s_1 s_2}(x)$ in the canonical ensemble, it is easy to see that the only modifications needed to this equation due to the charge-ordering constraint and the hard cores are that

$$\rho_{s_1 s_2(-s_2)}(x_1, x_2, x_2-\sigma)-\rho_{s_1 s_2(-s_2)}(x_1, x_2, x_2+\sigma) \qquad (2.47)$$

must be added to the r.h.s., and the smoothed potential $v(x)$ replaced by $v_c(x)$ [recall (2.11b)]. The definition of $Q_{s_1}(x_1 \mid x_3)$ remains that given by (2.46c) and the integration over x_3 remains extended over the whole real line. This is a consistent prescription, as the r.h.s. of (2.46a) can then be expressed entirely in terms of full distribution functions which vanish for configurations with $|x_i - x_j| < \sigma$. However, Eq. (2.46a) modified by the hard-core terms (2.47) is now valid for $|x_{12}| > \sigma$ [notice that the l.h.s. of (2.46a) diverges at $|x_{12}| = \sigma$].

Adding (2.47) to the r.h.s. of (2.46a), multiplying both sides by $q^2 s_1 s_2$, summing over $s_1, s_2 = \pm$, and using the definitions (2.24) and (2.46c) gives that the BGY equation for $C(x)$ in the charge-ordered system with hard cores is

$$\frac{\partial}{\partial x_2}(C(x_{12}) - 2q^2\rho\delta(x_{12}))$$

$$= 2\Gamma\rho \int_{-\infty}^{\infty} dx_3\, F_c(x_{32})\, C(x_{13}) + \Gamma q \int_{-\infty}^{\infty} dx_3\, F_c(x_{32})\, R(x_1, x_2, x_3)$$

$$+ qA(x_1, x_2) \tag{2.48a}$$

where

$$R(x_1, x_2, x_3) = \sum_{s_1, s_2, s_3 = \pm} q s_1 s_3 \rho^T_{s_1 s_2 s_3}(x_1, x_2, x_3)$$

$$+ q \sum_{s_1 s_2} \rho^T_{s_1 s_2}(x_1, x_2)\, \delta(x_3 - x_1) \tag{2.48b}$$

$$\doteq \sum_{s_2, s_3 = \pm} s_3 \rho_{s_2 s_3}(x_2, x_3)\, Q_{s_2 s_3}(x_2, x_3 \mid x_1) - \frac{2\rho}{q} C(x_{31}) \tag{2.48b'}$$

[the symbol $\doteq$ denotes that (2.48b') gives the same value as (2.48b) when substituted in (2.48a); see the sentence below (2.48d)] and

$$A(x_1, x_2) = B(x_1, x_2, x_2 - \sigma) - B(x_1, x_2, x_2 + \sigma) \tag{2.48c}$$

with

$$B(x_1, x_2, x) = \rho_{+-}(x_2, x)\, Q_{+-}(x_2, x \mid x_1) - \rho_{-+}(x_2, x)\, Q_{-+}(x_2, x \mid x_1)$$

$$- q(\delta(x_1 - x_2) - \delta(x_1 - x))(\rho_{+-}(x_2, x) + \rho_{-+}(x_2, x)) \tag{2.48d}$$

In deriving (2.48) we have used the equation

$$\int_{-\infty}^{\infty} dx\, F_c(x) \sum_{s_2, s_3 = \pm} s_2 s_3 \rho_{s_2 s_3}(0, x) = 0$$

which is a consequence of the first factor in the integrand being odd while the second factor is even.

In preparation for analyzing (2.48a) we note that on the l.h.s. we can make the replacement

$$\frac{\partial}{\partial x_2} \mapsto -\frac{\partial}{\partial x_1} \tag{2.49}$$

without changing its value. Similarly, on the r.h.s. we can write

$$F_c(x_{32}) = -\frac{\partial}{\partial x_2} v_c(x_{23}) = \frac{\partial}{\partial x_3} v_c(x_{32}) \tag{2.50}$$

The coordinate x_2 is now not involved in any operation, so for convenience it can be set equal to zero. Let us now proceed to transform the modified BGY equation into Fourier space. To do this we multiply both sides of the x_2-independent form of (2.48a) by e^{ikx_1} and integrate over x_1 with the condition $|x_1| > \sigma$. On the l.h.s. we have

$$\left(\int_{-\infty}^{-\sigma} + \int_{\sigma}^{\infty}\right) dx_1\, e^{ikx_1} \left\{-\frac{\partial}{\partial x_1}[C(x_1) - 2q^2\rho\delta(x_1)]\right\} \tag{2.51a}$$

which after integration by parts becomes

$$-2i\sin(k\sigma)\, C(\sigma) + ik[\tilde{C}(k) - 2q^2\rho] \tag{2.51b}$$

On the r.h.s. we can extend the integration to the entire real line because this side vanishes for $|x_1| < \sigma$ and remains finite at $|x_1| = \sigma$. Now it follows from (2.48b) that $R(x_1, x_2, x_3)$ is a fully truncated quantity and thus the integral involving R on the r.h.s. of (2.48a) is absolutely convergent. The integral over x_1 of this term can therefore be done before the integral over x_3. Doing this and using the convolution theorem in the first term gives that the Fourier transform of the r.h.s. equals

$$-2\Gamma\rho ik\tilde{v}_c(k)\, \tilde{C}(k) + \Gamma q \int_{-\infty}^{\infty} dx_3 \frac{\partial v_c(x_3)}{\partial x_3} \tilde{R}(k; x_3) + q\tilde{A}(k) \tag{2.52a}$$

where, from (2.48b′) and (2.48c),

$$\tilde{R}(k; x_3) = \sum_{s_2, s_3 = \pm} s_3 \rho_{s_2 s_3}(0, x_3)\, \tilde{Q}_{s_2 s_3}(0, x_3 \mid k) - \left(\frac{2\rho}{q}\right) e^{ikx_3}\, \tilde{C}(k) \quad (2.52b)$$

and

$$\begin{aligned} q\tilde{A}(k) = {} & 2iq^2 \sin(k\sigma)[\rho_{+-}(\sigma) + \rho_{-+}(\sigma)] \\ & + q\rho_{-+}(\sigma)[\tilde{Q}_{+-}(0, -\sigma \mid k) + \tilde{Q}_{-+}(0, \sigma \mid k)] \\ & - q\rho_{+-}(\sigma)[\tilde{Q}_{+-}(0, \sigma \mid 0) + \tilde{Q}_{-+}(0, -\sigma \mid x_1)] \end{aligned} \quad (2.52c)$$

[in deriving (2.52c) we have used the symmetry $\rho_{+-}(x) = \rho_{-+}(-x)$, where $\rho_{s_1 s_2}(x) := \rho_{s_1 s_2}(0, x)$].

To obtain the desired sum rule, we consider the leading-order small-k behavior of (2.51) and (2.52). The term independent of k on both sides vanishes due to sum rules for the perfect screening of an internal charge:

$$\tilde{C}(0) = \int_{-\infty}^{\infty} dx\, C(x) = 0 \quad (2.53a)$$

and

$$\tilde{Q}_{s_a s_b}(x_a, x_b; 0) := \int_{-\infty}^{\infty} dx\, Q_{s_a s_b}(x_a, x_b \mid x) = 0 \quad (2.53b)$$

which are expected to be true in all phases of Coulomb systems.[17] The leading small-k term is thus proportional to k. From (2.51), on the l.h.s. it is given by

$$-2ik\sigma C(\sigma) - 2ikq^2\rho \quad (2.54a)$$

while from (2.52) the leading small-k terms on the r.h.s. are

$$\begin{aligned} & -2\Gamma\rho ik\tilde{v}_c(k)\, \tilde{C}(k) + ik\Gamma q \sum_{s_2, s_3 = \pm} s_3 \int_{-\infty}^{\infty} dx_3 \frac{\partial v_c(x_3)}{\partial x_3} \rho_{s_2 s_3}(0, x_3)\, p_{s_2 s_3}(0, x_3) \\ & + 2iq^2 k\sigma[\rho_{+-}(\sigma) + \rho_{-+}(\sigma)] + ikq\rho_{-+}(\sigma)[p_{+-}(0, -\sigma) + p_{-+}(0, \sigma)] \\ & - ikq\rho_{+-}(\sigma)[p_{+-}(0, \sigma) + p_{-+}(0, -\sigma)] \end{aligned} \quad (2.54b)$$

where

$$p_{s_1 s_2}(0, x_3) := \int_{-\infty}^{\infty} dx\, x Q_{s_1 s_2}(0, x_3 \mid x) \quad (2.54c)$$

Let us now equate (2.54a) and (2.54b). The resulting equation can be simplified by recalling that $\rho_{++}(x)=0$ for all $|x|<2\sigma$, and so

$$C(\sigma)= -q^2[\rho_{+-}(\sigma)+\rho_{-+}(\sigma)]$$

Furthermore, $\tilde{v}_c(k)=\pi/|k|$, and by symmetry $p_{++}(x)=p_{--}(x)=0$ for all x in any phase. We thus obtain the sum rule

$$\lim_{k\to 0}\frac{\pi\beta\tilde{C}(k)}{|k|}=1+\frac{\beta q}{2\rho}\int dx\,\frac{\partial v_c}{\partial x}\,[\rho_{-+}(x)\,p_{-+}(x)-\rho_{+-}(x)\,p_{+-}(x)]$$
$$+\frac{1}{q\rho}[\rho_{-+}(\sigma)\,p_{-+}(\sigma)-\rho_{+-}(\sigma)\,p_{+-}(\sigma)] \qquad (2.55)$$

which is to obeyed in all phases. However, we have commented in Section 1.1 that for the charge-ordered system $C(r)$ is expected to exhibit a decay which is $o(1/r^2)$, except possibly on the phase boundary. Such a decay implies $\tilde{C}(k)=o(|k|)$ and thus the l.h.s. of (2.55) vanishes, leaving us with a simplified form of the sum rule which is to be obeyed by the system in all phases except possibly on the phase boundary. Note in particular that the dipole moment $p_{+-}(x)$ is nonzero in both phases. In ref. 9 it was claimed that $p_{+-}(x)$ as calculated from the exact results at $\Gamma=1$ vanished. However, in reviewing that calculation, an error has been found and the correct conclusion is that $p_{+-}(x)$ is nonzero at $\Gamma=1$, in agreement with the general property.

2.4.1. Verification of the Sum Rule for the $\Gamma=0$ Reference System. We have shown in Section 2.1 that the $\Gamma=0$ reference system perfectly screens an internal charge. Since the perfect screening properties (2.53) are the two fundamental assumptions in deriving the sum rule (2.55) from the BGY equation (2.48a), we therefore expect the sum rule to be satisfied in the reference system. Let us check this property.

The reference system of Section 2.1 consists only of point particles, so the hard-core width in (2.55) needs to be taken to zero. Also, the final terms on the r.h.s. can be simplified due to the symmetries $\rho_{+-}(x)=\rho_{-+}(x)$, $p_{+-}(x)= -p_{-+}(x)$, and the pair potential in the reference system is identically zero. In these circumstances the sum rule (2.55) reads

$$0=1-\frac{2}{q\rho}\rho_{+-}(0)\,p_{+-}(0^+) \qquad (2.56)$$

[$p_{+-}(x)$ is discontinuous at the origin, so it is necessary to specify its value on one side].

From (2.9), $\rho_{+-}(0)=2\rho^2$. It remains to calculate $p_{-+}(x)$. From the definitions (2.54c) and (2.46c) we must first calculate $\rho^{(3)}_{+-+}(x_1, x_2, x_3)$ and $\rho^{(3)}_{+--}(x_1, x_2, x_3)$, which can be done by an appropriate extension of the method given in Section 2.1 to calculate $\rho^{(2)}_{+-}(x)$. We find

$$\rho_{+-+}(x_1, x_2, x_3) = \begin{cases} \rho^3(1+e^{-4\rho|x_3-x_2|}+e^{-4\rho|x_2-x_1|}+e^{-4\rho|x_3-x_1|}), & \text{for } x_1<x_2<x_3 \\ \rho^3(1+e^{-4\rho|x_3-x_2|}-e^{-4\rho|x_2-x_1|}-e^{-4\rho|x_3-x_1|}), & \text{for } x_1<x_3<x_2 \end{cases} \tag{2.57}$$

and

$$\rho_{+--}(x_1, x_2, x_3) = \begin{cases} \rho^3(1-e^{-4\rho|x_3-x_2|}+e^{-4\rho|x_2-x_1|}-e^{-4\rho|x_3-x_1|}), & \text{for } x_1<x_2<x_3 \\ \rho^3(1+e^{-4\rho|x_3-x_2|}+e^{-4\rho|x_2-x_1|}+e^{-4\rho|x_3-x_1|}), & \text{for } x_2<x_1<x_3 \end{cases} \tag{2.58}$$

Using (2.57), (2.58), and (2.9) to form $Q_{+-}(x_1, x_2 \mid x)$ as given by (2.46c), we can check the perfect screening property

$$\int_{-\infty}^{\infty} dx\, Q_{+-}(x_1, x_2 | x) = 0 \tag{2.59}$$

which together with (2.53a) is a fundamental assumption in the derivation of (2.53). We can also calculate the first moment of Q_{+-} to obtain

$$\rho_{+-}(x)\, p_{+-}(x) = \pm\frac{q\rho}{2} e^{-4\rho|x|} \tag{2.60}$$

where the positive sign is to be taken for $x>0$, while the negative sign holds for $x<0$.

From (2.60) we see that indeed the sum rule (2.55) is obeyed for the $\Gamma=0$ reference system.

3. THE SYSTEM WITHOUT CHARGE ORDERING

3.1. Decay of Correlations in the Conductive Phase

For $\Gamma<2$ the two-component log-gas without charge ordering is expected to perfectly screen an infinitesimal charge density. As noted in Section 1.2, a linear response argument[8] then leads to the sum rule (1.4b) for the asymptotic behavior of the charge–charge correlation. In this subsection we will note another derivation of (1.4b), using a new method due to Jancovici,[18] which is based on a linear response argument and macroscopic electrostatics. In ref. 18 the conductive phase is characterized by the applicability of macroscopic electrostatics at large length scales in the system. The log-gas in a conductive phase is thus considered as an infinite metal line obeying the laws of two-dimensional electrostatics. When combined with a linear response relation describing the change in the potential due to the addition of an external charge, this characterization implies (1.4b). Furthermore, it is clear that (1.4b) only applies to the leading nonoscillatory term of the charge–charge correlation only: the use of macroscopic electrostatics implies the charge density must be smoothed over some microscopic distance and thus the oscillatory terms averaged to zero.

3.1.1. Abe–Meeron-Type Resummations. In three-dimensional classical Coulomb systems a systematic way to study the high-temperature conductive phase is via the Abe–Meeron[12] diagrammatic expansion of the Ursell function $h_{s_1 s_2}(r)$ $[:=\rho^T_{s_1 s_2}(r)/\rho^2]$. This approach is also applicable to the present system.

Briefly, let us recall (see ref. 19 for a detailed recent review of the method, and an extension to quantum systems) that $h_{s_1 s_2}(r)$ is given by the sum of all the Mayer graphs built with the Mayer bonds

$$f_{s_1 s_2}(r) = \exp(-\Gamma v_{s_1 s_2}(r)) - 1$$

(here the potential $v_{s_1 s_2}$ corresponds to the regularized logarithmic Coulomb potential between unit charges of signs s_1 and s_2). Once the chain summations have been performed, the above set of Mayer graphs is exactly transformed into an a new set of "prototype" graphs Π with the same topological structure and two kinds of resummed bonds:

(i) The Debye-like bond $\phi_{s_1 s_2}(r)$ defined as the sum of all the convolution chains built with the Coulomb potential $s_1 s_2 v_c(r)$, and such that

$$\tilde{\phi}_{s_1 s_2}(k) = -\frac{\Gamma s_1 s_2 \tilde{v}_c(k)}{1 + 2\Gamma\rho\tilde{v}_c(k)}$$

(convolutions of this bond are to be excluded).

(ii) The resummed bond f_R:

$$f_R(r) = \exp[-\Gamma(v_{s_1 s_2}(r) - s_1 s_2 v_c(r)) + \phi_{s_1 s_2}(r)] - 1 - \phi_{s_1 s_2}(r)$$

At large distances, $v_{s_1 s_2}(r) - s_1 s_2 v_c(r)$ vanishes and so $f_R(r)$ behaves as $\phi^2_{s_1 s_2}(r)/2$. Now, since $\tilde{v}_c(k) \sim \pi/|k|$ for small $|k|$ we deduce from the definition of the Debye-like bond that $\phi_{s_1 s_2}(r)$ decays as $1/r^2$ for large r. Therefore all the resummed bonds (i) decay at least as $1/r^2$, while the resummed bonds (ii) decays as $1/r^4$; the Ursell functions $h_{s_1 s_2}(r)$ therefore exhibit a $1/r^2$ decay in the conductive phase. This is to be contrasted to the $1/r^4$ decay found in the high-temperature phase of the system with charge ordering, where the Abe–Meeron diagrammatics does not apply due to the presence of a many-body potential inducing the charge-ordering constraint.

3.2. Decay of Correlations in the Dipole Phase

Our objective in this subsection is to use low-fugacity expansions to deduce the large-r decay of the truncated two-particle distributions and the charge–charge correlation for $\Gamma \geqslant 2$. Since in the system without charge ordering $\rho_{++}(r) = \rho_{--}(r)$ and $\rho_{+-}(r) = \rho_{-+}(r)$ it suffices to consider $\rho^T_{++}(r)$ and $\rho^T_{+-}(r)$. We will study in detail the terms of $O(\zeta^4)$ since, as is argued below, the same asymptotic decay is expected of terms of higher order in ζ.

At order ζ^4 the low-fugacity expansions of the two-particle correlations can be calculated from (2.21), (2.22), and the explicit forms of the partition functions which occur therein. We find

$$\rho^{T(4)}_{++}(r) = \zeta^4 \int^* dx_1\, dx_2 \left[\frac{1}{2} \left| \frac{r(x_2 - x_1)}{(x_2 - r)(x_1 - r)\, x_1 x_2} \right|^\Gamma - S \frac{1}{|x_1|^\Gamma\, |r - x_2|^\Gamma} \right] \tag{3.1}$$

and

$$\rho^{T(4)}_{+-}(r) = \zeta^4 \int^* dx_1\, dx_2$$
$$\times \left[S \left| \frac{x_1(x_2 - r)}{r(x_2 - x_1)(x_1 - r)\, x_2} \right|^\Gamma - \frac{1}{r^\Gamma\, |x_1 - x_2|^\Gamma} - S \frac{1}{|x_2|^\Gamma\, |x_1 - r|^\Gamma} \right] \tag{3.2}$$

where

$$Sf(x_1, x_2) := \tfrac{1}{2}[f(x_1, x_2) + f(x_2, x_1)] \tag{3.3a}$$

and the notation * denotes the region of integration

$$\{(x_1, x_2) \in R^2\colon |x_1 - x_2| > \sigma,\ |x_1|, |x_2| > \sigma,\ |x_1 - r|, |x_2 - r| > \sigma\} \tag{3.3b}$$

From these expansions, and (2.24), we have

$$C^{(4)}(r) = q^2\zeta^4 \int^* dx_1\, dx_2 \left[\left| \frac{r(x_2 - x_1)}{(x_2 - r)(x_1 - r)\, x_1 x_2} \right|^{\Gamma} - 2S \left| \frac{x_1(x_2 - r)}{r(x_2 - x_1)(x_1 - r)\, x_2} \right|^{\Gamma} + \frac{2}{r^{\Gamma}\, |x_1 - x_2|^{\Gamma}} \right] \tag{3.4}$$

In Appendix B a method of analysis of the larger-r behavior of the integral (3.4) is given. We find that for $2 < \Gamma < 4$ the region of the integrand which gives the leading-order contribution comes from the physical configurations α of Fig. 5 in Appendix B, while for $\Gamma > 4$ the configurations β of Fig. 5 in Appendix B give the leading-order behavior. Furthermore, this remains true of the integrals (3.1) and (3.2). Here we will calculate the contribution to the large-r behavior of the integrals in (3.1) and (3.2) from these configurations.

For the configurations α of Fig. 5, which give the leading large-r behavior for $2 < \Gamma < 4$, we can replace $|x_1 - r|^{\Gamma}$ and $|x_2 - r|^{\Gamma}$ by r^{Γ} in (3.1) and (3.2), provided we also multiply by a factor of 2 due to the invariance of the integrand under the mappings $x_2 - r \mapsto x_2$, $x_1 - r \mapsto x_1$. This gives

$$\rho^{T(4)}_{++}(r) \sim \frac{2\zeta^4\sigma^2}{(\sigma r)^{\Gamma}} \int^* dx_1\, dx_2 \left[\frac{1}{2} \frac{|x_2 - x_1|^{\Gamma}}{|x_1|^{\Gamma}\, |x_2|^{\Gamma}} - S \frac{1}{|x_1|^{\Gamma}} \right] \tag{3.5}$$

and

$$\rho^{T(4)}_{+-}(r) \sim \frac{2\zeta^4\sigma^2}{(\sigma \tau)^{\Gamma}} \int^* dx_1\, dx_2 \left[S \frac{|x_1|^{\Gamma}}{|x_2 - x_1|^{\Gamma}\, |x_2|^{\Gamma}} - \frac{1}{|x_1 - x_2|^{\Gamma}} - S \frac{1}{|x_1|^{\Gamma}} \right] \tag{3.6}$$

where * now denotes the region of integration

$$\{(x_1, x_2) \in R\colon |x_1 - x_2| > 1,\ |x_1|,\ |x_2| > 1\} \tag{3.7}$$

and we have removed the σ dependence from the integrals by changing variables $x_1 \mapsto \sigma x_1$, $x_2 \mapsto \sigma x_2$. Note that the integrals in (3.5) and (3.6) are conditionally convergent: for fixed x_1 (or x_2) one must add the contributions of positive and negative x_2 (or x_1).

For $\Gamma > 4$, when the configurations β of Fig. 5 give the dominant contribution to the asymptotics, the integrals (3.1) and (3.2) are analyzed by expanding the integrands for x_1 near 0 and x_2 near r (again we need to multiply by a factor of 2 to account for the symmetry in the interchange of x_1 and x_2). This gives

$$\rho^{T(4)}_{++}(r) \sim \int^* \frac{du\, dv}{|uv|^{\Gamma}} \left[\Gamma A(u, v; r) + \frac{\Gamma^2}{2} \left(\frac{uv}{r^2} \right)^2 \right] \tag{3.8}$$

and

$$\rho_{+-}^{T(4)}(r) \sim \int^{*} \frac{du\,dv}{|uv|^{\Gamma}}\left[-\Gamma A(u, v; r)+\frac{\Gamma^{2}}{2}\left(\frac{uv}{r^{2}}\right)^{2}\right] \tag{3.9}$$

where

$$A(u, v; r) := \frac{uv}{r^{2}}+\frac{1}{3}\left[\left(\frac{u-v}{r}\right)^{3}+\left(\frac{v}{r}\right)^{3}-\left(\frac{u}{r}\right)^{3}\right]$$
$$-\frac{1}{4}\left[\left(\frac{u-v}{r}\right)^{4}-\left(\frac{v}{r}\right)^{4}-\left(\frac{u}{r}\right)^{4}\right] \tag{3.10a}$$

and * denotes the integration region

$$\{(u, v) \in R^{2}\colon |u| > \sigma, |v| > \sigma\} \tag{3.10b}$$

In performing the integrations, only those terms in the integrand even in u and v survive, so (3.8) and (3.9) simplify to give

$$\rho_{++}^{T(4)}(r) \sim \zeta^{4}\,\frac{2\Gamma(\Gamma-3)}{r^{4}}\left(\frac{1}{(\Gamma-3)\,\sigma^{\Gamma-3}}\right)^{2} \tag{3.11}$$

and

$$\rho_{+-}^{T(4)}(r) \sim \zeta^{4}\,\frac{2\Gamma(\Gamma+3)}{r^{4}}\left(\frac{1}{(\Gamma-3)\,\sigma^{\Gamma-3}}\right)^{2} \tag{3.12}$$

The formulas (3.8) and (3.9) have some similarities and some differences in relation to the analogous results for the 2dCG [ref. 7, Eq. (3.17)]. One similarity is that the $O(1/r^{4})$ terms proportional to Γ^{2} result from squaring the dipole–dipole potential uv/r^{2} between the neutral clusters which form the dominant configurations. Another similarity is that $A(u, v; r)$ is the multipole expansion (appropriately truncated) of the potential between the neutral clusters, and that all terms odd in the mappings $v \mapsto -v$ or $u \mapsto -u$ vanish. A crucial difference is that in one space dimension the log-potential is not harmonic, so unlike the two-dimensional case, the integration over $A(u, v; r)$ does not vanish. The first nonvanishing term is the quadrupole–quadrupole potential, which has the same $O(1/r^{4})$ decay as the square of the dipole–dipole interaction. This term gives a contribution of the same magnitude to both (3.8) and (3.9) but of opposite sign.

At the coupling $\Gamma=4$ the leading-order behavior of $\rho_{++}^{T(4)}(r)$ is given by the sum of (3.5) and (3.11), and the leading-order behavior of $\rho_{+-}^{T(4)}(r)$ is given by the sum of (3.6) and (3.12). At the coupling $\Gamma=2$ all regions of integration analyzed in Appendix B were bounded by an $O(1/r^2)$ decay. However, it is possible to evaluate the integrals in (3.1) and (3.2) directly by using partial fractions. This is done below, where it is shown that

$$\rho_{++}^{T(4)}(r) \sim \frac{2\pi^2\zeta^4}{3r^2} \tag{3.13}$$

and

$$\rho_{+-}^{T(4)}(r) \sim -\frac{2\pi^2\zeta^4}{3r^2} \tag{3.14}$$

In fact these results are precisely the values given by (3.5) and (3.6) at $\Gamma=2$; the coefficient of the $1/r^\Gamma$ term is therefore continuous as $\Gamma \to 2^+$.

For general $n>2$ the large-r behavior of $\rho_{++}^{T(2n)}(r)$ and $\rho_{+-}^{T(2n)}(r)$ can be deduced by hypothesizing the generalization of the configurations α and β of Fig. 5, which give the leading large-r contribution, for $2<\Gamma<4$ and $\Gamma>4$, respectively, to the integral expressions for $\rho_{++}^{T(4)}(r)$ and $\rho_{+-}^{T(4)}(r)$. The obvious generalization of configuration α, which should give the leading large-r expansion for $2<\Gamma<4$, has $n-2-p$ mobile particles about one root particle and the remaining $n+p$, with $0 \leqslant p \leqslant n-2$, mobile particles about the other root particle. The mobile particles are to be distributed so that the total charge of the cluster about and including one root particle is $+q$, while the total charge about and including the other root particle is $-q$. At large distance the effective potential is thus $q^2 \log r$, which gives the required behavior as

$$\rho_{++}^{T(2n)}(r) \sim \frac{A_{++}^{(2n)}(\Gamma)}{r^\Gamma}\zeta^{2n} \tag{3.15}$$

and

$$\rho_{+-}^{T(2n)}(r) \sim \frac{A_{+-}^{(2n)}(\Gamma)}{r^\Gamma}\zeta^{2n} \tag{3.16}$$

This is the same order decay as exhibited by (3.5) and (3.6) in the case $n=2$, and furthermore the amplitudes could, in principle, be given integral representations analogous to those for $n=2$.

In particular, the coefficients in (3.15) and (3.16) should be finite as $\Gamma \to 2^+$. This would imply that $C(r)$ as calculated from a resummation of (3.15) and (3.16) would exhibit the same power-law decay $O(1/r^\Gamma)$, independent of the fugacity ζ. We therefore expect that the critical line separating the conductive and insulator phases to be at $\Gamma=2$, independent of the fugacity, which is contrary to what we have found for the charge-ordered system, but in agreement with previous studies.[3, 20] Thus the nested dipoles of Section 2.2.3 do not give the dominant contribution to $C(r)$ in the system without charge ordering. In fact the analysis in Appendix B shows that the leading contribution of nested dipoles cancels out when the charge ordering constraint is removed.

The generalizations of the configuration β of Fig. 5, which should give the leading behavior for $\Gamma > 4$, are neutral clusters about and including the root particles. The potential V_{2N} between these neutral clusters can be expanded as

$$V_{2N} = W_0 + W_r + U_{0r}$$

where W_0 and W_r are the electrostatic energies of the neutral clusters about 0 and r, respectively. For large r, the potential U_{0r} can be written as a multipole expansion in r. Analogous to the case $n = 1$, there will be two classes of terms contributing to the final leading-order expansions

$$\rho_{++}^{T(2n)}(r) \sim \frac{\zeta^{2n} A_{++}^{(2n)}(\Gamma)}{r^4} \tag{3.17}$$

and

$$\rho_{+-}^{T(2n)}(r) \sim \frac{\zeta^{2n} A_{+-}^{(2n)}(\Gamma)}{r^4} \tag{3.18}$$

3.2.1. Decay at $\Gamma = 2$. Here we will show how (3.13) and (3.14) can be derived. Consider for definiteness (3.13). From the Cauchy identity

$$\left(\frac{r(x_2 - x_1)}{(x_2 - r)(x_1 - r)\, x_1 x_2}\right)^2 = \left(\det \begin{bmatrix} 1/x_1 & 1/(x_1 - r) \\ 1/x_2 & 1/(x_2 - r) \end{bmatrix}\right)^2$$

we see that at $\Gamma = 2$, (3.1) can be simplified to read

$$\rho_{++}^{T(4)}(r) = -\zeta^4 \int^* dx_1\, dx_2 \frac{1}{(x_2 - r)(x_1 - r)\, x_1 x_2} \tag{3.19}$$

where the integration domain * is that specified by (3.3b). The integration over x_2 can now be performed using the simple partial fraction expansion

$$\frac{1}{(x_2 - r)\, x_2} = \frac{1}{r}\left(\frac{1}{x_2 - r} - \frac{1}{x_2}\right) \tag{3.20}$$

We must consider separately the regions $-\infty < x_1 < -2\sigma$, $-2\sigma < x_1 < \sigma$, $\sigma < x_1 < 2\sigma$, and $2\sigma < x_1 < 1/2$ [for $x_1 > 1/2$ an identical contribution to (3.19) is obtained]. This gives, after changing variables $x_1 \mapsto ru$,

$$\rho_{++}^{T(4)}(r) \sim -\frac{2\zeta^4}{r^2}\left\{\left(\int_{-\infty}^{-2\sigma/r} + \int_{2\sigma/r}^{1/2}\right)\frac{du}{(u-1)u}\left[\log\left|\frac{u+\sigma/r}{u-\sigma/r}\right|\right]\right.$$
$$+\int_{-2\sigma/r}^{-\sigma/r}\frac{du}{(u-1)u}\left[\log\left|\frac{\sigma/r}{u-\sigma/r}\right| - \log\left|\frac{-1+\sigma/r}{u-1-\sigma/r}\right|\right]$$
$$\left.+\int_{\sigma/r}^{2\sigma/r}\frac{du}{(u-1)u}\left[\log\left|\frac{u+\sigma/r}{\sigma/r}\right| - \log\left|\frac{u-1+\sigma/r}{-1-\sigma/r}\right|\right]\right\} \tag{3.21}$$

where we have ignored an additive term $\log[(1+\sigma/r)/(1-\sigma/r)]$ in each integrand since it decays as $r \to \infty$. In fact the only portion of the integrand which does not decay as $r \to \infty$ comes form the region $u = O(\sigma/r)$. Setting $u = x\sigma/r$ and keeping only those terms which are nonzero gives

$$\rho_{++}^{T(4)}(r) \sim \frac{4\zeta^4}{r^2}\left(\int_1^2 dx\,\frac{1}{x}\log(1+x) + \int_2^\infty dx\,\frac{1}{x}\log\frac{x+1}{x-1}\right) \tag{3.22}$$

Evaluating these integrals gives the result (3.13).

3.3. A Sum Rule from the BGY Equation for $C(x)$

Our objective in this subsection is to derive sum rules analogous to (2.55) for the system without charge ordering. We will consider first the case when the logarithmic attraction between opposite charges is smoothly regularized. In this case we will provide an illustration of the sum rule using a solvable "parallel line" model at $\Gamma = 2$. The other case to be considered is the hard-core regularization used in the above subsection.

Let us denote the smoothly regularized potentials by $v_{s_1 s_2}(x)$ and define the corresponding force by

$$F_{s_1 s_2}(x) = -\frac{d}{dx} v_{s_1 s_2}(x) \tag{3.23}$$

(using this notation rather than that of Section 2.4 allows for the possibility of different regularization between like and unlike species). The BGY equation for $\rho_{s_1 s_2}(x_{12})$ is then given by (2.46a) with $s_i s_j F(x_{ij})$ therein replaced by $F_{s_i s_j}(x_{ij})$. Assuming the symmetry

$$F_{s_1 s_2}(x) = F_{(-s_1)(-s_2)}(x)$$

it is straightfoward to show from the BGY equation for $\rho_{s_1 s_2}(x_{12})$ that the BGY equation for the charge–charge correlation in this system is [recall the derivation of (2.48a) from (2.46a)]

$$\frac{\partial}{\partial x_2}[C(x_{12})-2q^2\rho\delta(x_{12})]$$

$$=\Gamma\rho\int_{-\infty}^{\infty}dx_3\,[F_{++}(x_{32})-F_{+-}(x_{32})]\;C(x_{13})$$

$$+2\Gamma q\int_{-\infty}^{\infty}dx_3\,F_{++}(x_{32})[\rho_{++}(x_{32})\;Q_{++}(x_2,x_3\mid x_1)-(\rho/2q)\;C(x_{31})]$$

$$-2\Gamma q\int_{-\infty}^{\infty}dx_3\,F_{+-}(x_{32})[\rho_{+-}(x_{32})\;Q_{-+}(x_2,x_3\mid x_1)-(\rho/2q)\;C(x_{31})] \tag{3.24}$$

We now follow the procedure detailed in Section 2.4 to study the small-k behavior of the Fourier-transformed version of (3.24). The final result is the sum rule

$$\lim_{k\to 0}\frac{\pi\beta\tilde{C}(k)}{|k|}=1-\frac{\beta q}{\rho}\int_{-\infty}^{\infty}dx\,\frac{\partial v_{+-}(x)}{\partial x}\,\rho_{-+}(x)\;p_{-+}(x) \tag{3.25}$$

where the dipole moment $p_{-+}(x)$ is defined by (2.54c). In the distinct phases different terms in (3.25) have special values, which allows further simplification. In the conductor phase $\Gamma<2$ we expect $p_{-+}(x)=0$, which implies

$$\pi\beta\tilde{C}(k)/|k|=1 \tag{3.26}$$

This result is equivalent to (1.4b). On the other hand, in the insulator phase ($\Gamma>2$) the expected decay (1.4a) implies the l.h.s. of (3.25) vanishes and thus the dipole moment $p_{-+}(x)$ is nonzero.

In the case of a hard-core regularization of the logarithmic potential, working analogous to that given in Section 2.4 for the charge-ordered system shows that the sum rule (2.55) is still applicable. Without charge ordering the correlations obey

$$\rho_{+-}(x)\;p_{+-}(x)=-\rho_{-+}(x)\;p_{-+}(x)$$

so the sum rule reads

$$\lim_{k \to 0} \frac{\pi\beta\tilde{C}(k)}{|k|} = 1 - \frac{\beta q}{\rho} \int_{-\infty}^{\infty} dx \, \frac{\partial v_c}{\partial x} \, p_{+-}(x) \, p_{+-}(x)$$

$$- \frac{2}{q\rho} \, p_{+-}(\sigma) \, p_{+-}(\sigma) \tag{3.27}$$

In the insulator phase ($\Gamma > 2$) we expect the behavior (1.4a), so the l.h.s. of (3.27) will vanish. In the conductor phase ($\Gamma < 2$) the dipole moment $p_{+-}(x)$ should vanish and thus (3.26) is reclaimed.

3.3.1. Verification of the Sum Rule for a Parallel Line Model at $\Gamma = 2$. The 2dCG is exactly solvable at the coupling $\Gamma = 2$,[11] where the multiparticle distribution functions can be calculated. The method of exact calculation used in ref. 21 can be adapted to a model in which the negative charges (coordinates x_j, $j = 1, \ldots, N$) are confined to a line and the positive charges (coordinates X_j, $j = 1, \ldots, N$) are confined to another line, parallel to the first line and separated by a distance ε. The pair potential between opposite charges of unit strength is thus given by

$$v_{-+}(x - X) = \log[(x - X)^2 + \varepsilon^2]^{1/2} \tag{3.28}$$

while the pair potential between like charges is the Coulomb logarithmic potential. Notice that (3.28) is a smoothly regularized Coulomb potential. As such, the system should obey the sum rule (3.25). Let us verify this fact at the solvable coupling $\Gamma = 2$.

To verify (3.25) we need to evaluate $p_{-+}(x_1, x_2)$, which requires the two- and three-particle distributions. Using the technique of ref. 21, we can easily evaluate the general distribution function as

$$\rho_{+\cdots+-\cdots-}(X_1, \ldots, X_{n_1}; x_1, \ldots, x_{n_2}) \tag{3.29a}$$

$$= \det \begin{bmatrix} [G_{++}(X_j, X_k)]_{j,k=1,\ldots,n_1} & [G_{-+}(x_j, X_k)]_{j=1,\ldots,n_2;\, k=1,\ldots,n_1} \\ [G_{+-}(X_j, x_k)]_{j=1,\ldots,n_1;\, k=1,\ldots,n_2} & [G_{--}(x_j, x_k)]_{j,k=1,\ldots,n_2} \end{bmatrix} \tag{3.29b}$$

where

$$G_{++}(x) = G_{--}(x) = \int_0^{\infty} d\gamma \, \frac{e^{2\pi i x \gamma}}{1 + (1/2\pi\zeta)^2 \, e^{4\pi\varepsilon\gamma}} \tag{3.29c}$$

and

$$G_{-+}(x) = G_{+-}(x) = -\frac{1}{2\pi i\zeta} \int_0^\infty d\gamma \, \frac{e^{2\pi\varepsilon\gamma + 2\pi i x\gamma}}{1 + (1/2\pi\zeta)^2 \, e^{4\pi\varepsilon\gamma}} \tag{3.29d}$$

We want to use these formulas to evaluate $p_{+-}(x)$ as given by (2.54c). For this purpose we note from (2.46c) that

$$\begin{aligned} &\rho_{-+}(x_1, x_2)\, Q_{-+}(x_1, x_2; x) \\ &\quad = q(\rho^T_{+-+}(x, x_1, x_2) - \rho^T_{--+}(x, x_1, x_2) + \alpha(x; x_1, x_2)) \end{aligned} \tag{3.30a}$$

where

$$\begin{aligned} \alpha(x; x_1, x_2) = {}& \rho\rho^T_{+-}(x, x_1) + \rho\rho^T_{++}(x, x_2) - \rho\rho^T_{--}(x, x_1) - \rho\rho^T_{-+}(x, x_2) \\ &+ [\delta(x - x_2) - \delta(x - x_1)]\, \rho_{-+}(x_1, x_2) \end{aligned} \tag{3.30b}$$

Now it follows from the perfect screening sum rule (2.53a) for a single internal charge [which can be verified using (3.29)], and the dependence of the truncated two-particle distributions $\rho^T_{s_1, s_2}(x, x')$ on $|x - x'|$, that

$$\int_{-\infty}^{\infty} dx\, x\alpha(x; x_1, x_2) = q(x_2 - x_1)\, \rho^T_{-+}(x_1, x_2) \tag{3.31}$$

Next, to evaluate

$$\int_{-\infty}^{\infty} dx\, x(\rho^T_{+-+}(x, x_1, x_2) - \rho^T_{--+}(x, x_1, x_2)) \tag{3.32}$$

we note from (3.29) that

$$\begin{aligned} &\rho^T_{+-+}(x, x_1, x_2) - \rho^T_{--+}(x, x_1, x_2) \\ &\quad = 2\,\mathrm{Re}(G_{++}(x, x_2)\, G_{+-}(x_2, x_1)\, G_{-+}(x_1, x) \\ &\qquad - G_{++}(x, x_1)\, G_{+-}(x_1, x_2)\, G_{-+}(x_2, x)) \end{aligned} \tag{3.33}$$

which shows that it suffices to evaluate

$$\int_{-\infty}^{\infty} dx\, xG_{++}(r, x_2)\, G_{-+}(x_1, x) \tag{3.34}$$

This latter task can be carried out with the aid of the formula

$$\int_{-\infty}^{\infty} dx\, x e^{2\pi i x(y_1-y_2)}=\frac{1}{2\pi i}\frac{\partial}{\partial y_1}\delta(y_1-y_2) \tag{3.35}$$

and integration by parts. Once having evaluated (3.32), we substitute the result in the integral over r of (3.30a) together with (3.31) to conclude

$$\begin{aligned}\rho_{-+}(x_1,x_2)\,\rho_{-+}(x_1,x_2)=&\frac{iq\varepsilon}{(2\pi\zeta)^2}\left[\int_0^{\infty} dy\,\frac{e^{2\pi\varepsilon y+2\pi i y(x_2-x_1)}}{1+(1/2\pi\zeta)^2 e^{4\pi\varepsilon y}}\right.\\ &\left.\times\int_0^{\infty} dy_1\,\frac{e^{2\pi\varepsilon y_1+2\pi i y_1(x_2-x_1)}}{[1+(1/2\pi\zeta)^2 e^{4\pi\varepsilon y_1}]^2}-(x_1\leftrightarrow x_2)\right]\\ &+q(x_2-x_1)\,\rho^T_{-+}(x_1,x_2)\end{aligned} \tag{3.36}$$

With the result (3.36) and the formula

$$\frac{\partial v_{-+}(x)}{\partial x}=\frac{1}{2}\left(\frac{1}{x+i\varepsilon}+\frac{1}{x-i\varepsilon}\right)$$

for the force between opposite charges, all quantities in the integrand on the r.h.s. of the sum rule (3.25) are known. The integral can now be carried out by aid of the formula

$$\int_{-\infty}^{\infty} dx\,\frac{e^{2\pi i(y_1-y_2)x}}{x+i\varepsilon}=-2\pi i e^{2\pi(y_1-y_2)\varepsilon}\times\begin{cases}1, & y_1-y_2<0\\ 0, & \text{otherwise}\end{cases}$$

We find

$$\int_{-\infty}^{\infty} dx\,\rho_{-+}(x)\,\rho_{-+}(x)\frac{\partial v_{-+}(x)}{\partial x}=\frac{q\rho}{2}\left(1-\frac{1}{1+(1/2\pi\zeta)^2}\right) \tag{3.37}$$

which thus evaluates the r.h.s. of (3.25).

To evaluate the l.h.s. of (3.25), we note from exact formula (3.29) the large-x expansion

$$C(x)\sim-\frac{q^2}{2(\pi x)^2}\frac{1}{1+1/(2\pi\zeta)^2}$$

which implies

$$\lim_{k\to 0}\frac{\pi\beta\tilde{C}(k)}{|k|}\sim\frac{1}{1+1/(2\pi\zeta)^2} \tag{3.38}$$

Substituting (3.37) and (3.38) in (3.25), we see that the sum rule is indeed valid.

3.4. Decay of Three- and Four-Body Correlations in the Dipole Phase

The decay of the higher order correlations can be studied using the low-fugacity expansion method of Section 3.2. Alternatively, the decay of the three-body correlations can be deduced from knowledge of the decay of the two-body correlations and use of the BGY equation. Let us consider this latter method.

Suppose for definiteness that the logarithmic potential is smoothly regularized at the origin and $v_{s_1 s_2} = s_1 s_2 v$, so that the BGY equation (2.46a) applies. Making the changes (2.49) and (2.50) allows the Fourier transform with respect to x_1 of both sides of (2.46a) to be taken, with the result

$$-ik\tilde{\rho}^T_{s_1 s_2}(k) = s_1 s_2 \Gamma \int_{-\infty}^{\infty} dx \frac{\partial v(x)}{\partial x} \rho^T_{s_1 s_2}(x)\, e^{ikx} - \frac{s_1 s_2}{2} \Gamma \rho ik\tilde{v}(k)\, \tilde{C}(k)$$
$$+ s_2 \Gamma \int_{-\infty}^{\infty} dx_3 \frac{\partial v(x_3)}{\partial x_3} [\tilde{\rho}^T_{s_1 s_2 +}(k, 0, x_3) - \tilde{\rho}^T_{s_1 s_2 -}(k, 0, x_3)] \tag{3.39}$$

Now from (3.5) and (3.6), for $2 < \Gamma < 4$

$$\rho^T_{s_1 s_2}(x) \underset{x \to \infty}{\sim} \frac{1}{|x|^\Gamma} \qquad \text{and so} \quad \tilde{\rho}^T_{s_1 s_2}(k) \underset{k \to 0}{\sim} |k|^{\Gamma - 1} \tag{3.40}$$

(here and below, the numerical amplitudes in the asymptotic formulas are omitted). The first term on both sides of (3.39) is thus $O(|k|^\Gamma)$ [here we have also used $F(x) \sim 1/x$]. Consider the second term on the r.h.s. Since $v_c(k) = \pi/|k|$, nonanalytic behavior results from both the leading nonanalytic and leading analytic term in the expansion of $\tilde{C}(k)$:

$$\tilde{C}(k) \sim |k|^{\Gamma - 1} + k^2 \tag{3.41}$$

This term is of a lower order in k than the previous two considered and so must be balanced by the three-body term:

$$\int_{-\infty}^{\infty} dx_3 \frac{\partial v(x_3)}{\partial x_3} [\tilde{\rho}^T_{s_1 s_2 +}(k, 0, x_3) - \tilde{\rho}^T_{s_2 s_2 -}(k, 0, x_3)]$$
$$\sim \frac{s_1}{2} \rho i\pi \operatorname{sgn}(k)\, (|k|^{\Gamma - 1} + k^2) \tag{3.42}$$

Note that for $\Gamma > 3$ the $O(k^2)$ term is dominant. Taking the inverse transform suggests that for large $|x_1|$ and fixed x_3

$$\rho^T_{s_1 s_2 +}(x_1, 0, x_3) - \rho^T_{s_1 s_2 -}(x_1, 0, x_3) \sim \begin{cases} O\left(\dfrac{\operatorname{sgn}(x_1)}{|x_1|^\Gamma}\right), & 2 < \Gamma < 3 \\ O\left(\dfrac{1}{x_1^3}\right), & \Gamma > 3 \end{cases} \tag{3.43}$$

At $\Gamma = 3$, an extra logarithmic factor is expected.

The nonperturbative result (3.43) is an agreement with a low-fugacity analysis. In particular, the $O(1/x_1^3)$ behavior of $\rho^T_{s-+}(x_1, 0, x_3)$ at $O(\zeta^4)$, which is dominant for $\Gamma > 3$, originates from the configuration in which the mobile particle of charge $-s$ forms a dipole with the root charge at x_1. Explicitly, the first term of the multipole expansion of this charge configuration which gives a nonzero contribution to the cluster integral is

$$(u - x_1)^2 (x_3 - x_2) \frac{\partial^2}{\partial x_1^2} \frac{\partial}{\partial x_2} (-\log |x_1 - x_2|) \sim \frac{1}{x_1^3} \tag{3.44}$$

For the four-body correlations, an analysis of the BGY equation for the three-body correlations analogous to that given above leads to the conclusion that for all $\Gamma > 2$

$$\rho^T_{s_1(-s_1)s_2(-s_2)}(x_1, x_2, x_3, x_4) \sim \frac{1}{(x_1 - x_3)^2} \tag{3.45}$$

as $|x_1 - x_3| \to \infty$ with $|x_1 - x_2|$ and $|x_3 - x_4|$ fixed.

3.4.1. Exact Decay of a Three-Particle Correlation at $\Gamma = 4$. A version of the two-component log-gas system without charge ordering, in which the two species of charges are confined to separate, interlacing sublattices (lattice spacing τ), is exactly solvable at the isotherm $\Gamma = 4$.[10] In particular, the exact expression for the three-particle correlation involving opposite charges is[22]

$$\begin{aligned} &\rho^T_{+--}(n, m_1 - 1/2, m_2 - 1/2) \\ &\quad = \tfrac{1}{2} \operatorname{Tr}[B_{+-}(n, m_1 - 1/2)\, B_{--}(m_1 - 1/2, m_2 - 1/2)\, B_{-+}(m_2 - 1/2, n)] \\ &\qquad + \tfrac{1}{2} \operatorname{Tr}[B_{+-}(n, m_2 - 1/2)\, B_{--}(m_2 - 1/2, m_1 - 1/2)\, B_{-+}(m_1 - 1/2, n)] \end{aligned} \tag{3.46}$$

where

$$B_{+-}(n, m_1 - 1/2) = \begin{pmatrix} \beta_{13}(l_1) & \beta_{14}(l_1) \\ \beta_{14}(l_1) & \beta_{31}(l_1) \end{pmatrix}$$

$$B_{-+}(m_2 - 1/2, n) = \begin{pmatrix} \beta_{31}(l_2) & \beta_{14}(l_2) \\ \beta_{14}(l_2) & \beta_{13}(l_2) \end{pmatrix}$$

with

$$\beta_{13}(l) = \frac{\pi\zeta i}{\tau}(1+\xi)\int_0^1 dt\, \frac{e^{2\pi i l t}}{D(t)}$$

$$\beta_{14}(l) = \frac{2\pi^2\zeta}{\tau^2}(1-\xi)\int_0^1 dt\, \frac{e^{2\pi i l t}}{D(t)}\left(\frac{1}{2} - t\right)$$

$$\beta_{31}(l) = \frac{2\pi^3\zeta}{\tau^3 i}(1+\xi)\int_0^1 dt\, \frac{e^{2\pi i l t}}{D(t)}\, t(t-1)$$

$$D(t) := 1 - 16\xi(t^2 - t + 1/8) + \xi^2, \qquad \xi := \zeta^2\pi^4/\tau^4$$

$$l_1 := n - (m_1 - 1/2), \qquad l_2 := m_2 - 1/2 - n$$

and

$$B_{--}(m_1 - 1/2, m_2 - 1/2) = \begin{pmatrix} \beta_{11}(l_3) & \beta_{21}(l_3) \\ \beta_{12}(l_3) & \beta_{11}(l_3) \end{pmatrix}$$

with

$$\beta_{11}(l) = \int_0^1 dt\, \frac{e^{2\pi i l t}}{D(t)}\left[\xi^2 - 4\xi t(1-t) - 4\xi\left(t - \frac{1}{2}\right)^2\right]$$

$$\beta_{12}(l) = 4\left(\frac{\tau}{\pi}\right)\xi i\int_0^1 dt\, \frac{e^{2\pi i l t}}{D(t)}\left(t - \frac{1}{2}\right)$$

$$\beta_{21}(l) = \frac{16(\pi/\tau)\xi}{i}\int_0^1 dt\, \frac{e^{2\pi i l t}}{D(t)}\left(t - \frac{1}{2}\right) t(t-1)$$

$$l_3 := m_1 - m_2$$

We seek the asymptotic behavior of (3.46) for $|m_2| \to \infty$ with n and m_1 fixed. Now, straightforward integration by parts of the above integrals gives

$$\beta_{13}(l) \sim \frac{1}{l}, \qquad \beta_{31}(l) \sim \frac{1}{l^3}, \qquad \beta_{14}(l) \sim \frac{1}{l^2}$$

$$\beta_{11}(l) \sim \frac{1}{l^2}, \qquad \beta_{12}(l) \sim \frac{1}{l}, \qquad \beta_{21}(l) \sim \frac{1}{l^3}$$

and thus

$$B_{--}(m_1-1/2, m_2-1/2)\, B_{-+}(m_2-1/2, n) \sim \begin{pmatrix} 1/m_2^5 & 1/m_2^4 \\ 1/m_2^4 & 1/m_2^3 \end{pmatrix} \tag{3.47a}$$

$$B_{+-}(n, m_2-1/2)\, B_{--}(m_2-1/2, m_1-1/2) \sim \begin{pmatrix} 1/m_2^3 & 1/m_2^4 \\ 1/m_2^4 & 1/m_2^5 \end{pmatrix} \tag{3.47b}$$

Substituting (3.47) in (3.46) gives that

$$\rho^T_{+--}(n, m_1-1/2, m_2-1/2) \underset{m_2 \to \infty}{\sim} 1/m_2^3$$

in agreement with the prediction (3.43).

4. SUMMARY AND COMMENTS

We have undertaken a systematic study of properties of the particle and charge correlation functions in the two-dimensional Coulomb gas confined to a one-dimensional domain, with and without charge ordering. As a result some new properties have emerged.

Consider first the system with charge ordering. In the high-temperature phase we have found an $O(1/x^4)$ decay of the two-particle correlations as given by (2.18). The system does not screen an external charge density, as this would lead to (1.4b) and thus an $O(1/x^2)$ decay. However, the underlying reference system at $\Gamma=0$ does screen an internal charge. This allows an expansion of the large-x two-particle correlations to made about $\Gamma=0$ and leads to the $O(1/x^4)$ decay. For $\Gamma \to 2^+$, it was shown in Section 2.2 that the asymptotic charge–charge correlation can be resummed to all orders in the fugacity, and that the Kosterlitz renormalization equations result. The radius of convergence of the quantity Δ, (2.19), as given by (2.37) gives the dependence of the critical coupling on the fugacity, analogous to the situation with the resummed dielectric constant in the 2dCG.[7] Furthermore, the configurations contributing to the asymptotic charge–charge correlation were identified as nested dipoles, analogous to situation in the scaling region of the Kosterlitz–Thouless transition of the 2dCG. Away from criticality in the low-temperature phase the two-particle correlations have an $O(1/r^2)$ decay as given by (2.44). In a low-fugacity perturbation expansion this behavior can be understood as resulting from the dipole–dipole interaction between neutral clusters of particles about each root particle. In all phases the correlations must obey the sum rule (2.55) involving the dipole moment $p_{+-}(x)$ of the charge distribution induced by two fixed opposite charges. This sum rule is analogous to the sum rule (1.3b) for the 2dCG, which relates the dielectric constant to p_{-+}.

Now consider the system without charge ordering. In the high-temperature phase $\Gamma \leqslant 2$ the system is conductive and the charge correlation must exhibit the asymptotic behavior (1.4b). The low-fugacity analysis of the two-particle correlations given in Section 3.2 implies a $O(1/x^{\Gamma})$ decay for $1 \leqslant \Gamma < 4$ and an $O(1/x^4)$ decay for $\Gamma \geqslant 4$. In the context of the low-fugacity expansion this behavior results from configurations consisting of clusters about the root charges which have a charge imbalance equivalent to that of one particle dominating for $2 < \Gamma < 4$, while neutral clusters dominate for $\Gamma > 4$. Furthermore, for $\Gamma \to 2^+$ the coefficients of the low-fugacity expansion of the large-x two-particle correlations are all finite; there is no dominance of nested dipole configurations, which in fact cancel without charge ordering. The finiteness of the coefficients indicates that the phase transition occurs at $\Gamma = 2$ independent of the fugacity. With a smoothly regularized potential, we have presented the sum rule (3.25) involving the dipole moment $p_{+-}(x)$, and explicitly verified it on a solvable model at the coupling $\Gamma = 2$.

As far as large-distance behaviors of the internal correlations are concerned, the system with charge ordering is very similar to the familiar 2D Coulomb gas. Indeed, the charge correlation have a "fast" decay in the high-temperature phase ($1/|x|^4$ in comparison to an exponential decay in 2D), and as a density-dependent power law in the low-temperature phase ($1/|x|^{\Gamma\Delta}$ in comparison to $1/|r|^{\Gamma/\varepsilon}$ in 2D). The partial screening of a given pair (with sizes X in 1D and R in 2D) by smaller pairs takes the same mathematical form for both systems, because of the occurrence of the integrals

$$\int_{\sigma < x < X} dx \frac{d}{dx} \log |x| \tag{4.1a}$$

and

$$\int_{\sigma < r < R} d\mathbf{r} \, (\nabla \log r)^2 \tag{4.1b}$$

which both diverge logarithmically when X and R go to infinity. The integral (4.1a) arises from the dipole–dipole potential between the screening pair and the root charges. The resulting screening contribution is proportional to the average of the dipole carried by this pair, which does not vanish, by virtue of the charge ordering constraint.

Of course, the screening contribution associated with (4.1a) disappears in the system without charge ordering. The leading contribution of a screening pair then involves

$$\int_{\sigma < x < X} dx \left(\frac{d}{dx} \log |x| \right)^2 \tag{4.2}$$

which is linked to the fluctuations of the dipole–charge potential between this pair and the root charges. The integral (4.2) does converge in the limit $X \to \infty$, and consequently the collective effects do not affect the power which controls the algebraic decay of the charge correlations. The Kosterlitz–Thouless transition temperature is then expected to remain constant ($\Gamma = 2$) at low densities. This is contrary to the finding of a recent computer simulation,[23] where the transition coupling appeared to decrease with decreasing density. However, we believe these findings are linked to finite effects which become more and more important as the density decreases. Indeed, in the zero-density limit and with $1 < \Gamma < 2$, the proportion of free charges goes to zero (see, e.g., ref. 15), a behavior which is reminiscent of the short-range collapse at $\Gamma = 1$ for the system without hard-core regularization. In this regime of parameters, the finite systems considered in the numerical simulations are not efficient for perfectly screening external charges, because of the very small number of free charges which are available. A similar mechanism also occurs in 2D,[24] and even in 3D[25] for the restricted primitive model with $1/r$ interactions (in that case there is no Kosterlitz–Thouless transition). Eventually, we note that at high densities, despite difficult problems of sampling, the numerical simulations[23] are compatible with the theoretical predictions given here and in ref. 3.

Our predictions should be valid at low densities. At higher densities, the critical Kosterlitz–Thouless transition line in the (ρ, T) plane might bifurcate, at a tricritical point, into a first-order liquid–gas coecistence curve. In fact, for the 2D Coulomb gas, this has been observed in computer simulations by Caillol and Levesque.[26] More recently, an extended Debye–Hückel theory of Levin *et al.*[27] suggests a tricritical point at a much lower density. To our knowledge, such first-order transitions have not yet been observed for the present models in 1D. At a theoretical level, their study is beyond the scope and the methods of this paper. In particular, a correct description of eventual oscillatory correlations[28] beyond a Fisher–Widom line requires further resummations of the low-fugacity expansions combined with a detailed account of the short-range effects which depend on the hard-core regularization. At the moment, we note that for the model without charge ordering, the exact results for the solvable isotherm $\Gamma = 2$[10,11] seem to exclude the above bifurcation process.

APPENDIX A

Our objective in this appendix is to show how the integrals in (2.23) can be analyzed to determined the asymptotic charge density $C_\Delta^{(4)}(r)$. We recall $C_\Delta^{(4)}(r)$ is defined as the portion of $C(r)$ proportional to ζ^4 which contributes to the correct leading-order behavior of (2.9) in the limit $\Gamma \to 2^+$.

We will illustrate our method by calculating the contribution to $C_\Delta^{(4)}$ from $\rho_{++}^{(4)}(r)$. Scaling the integration variables by

$$y_1 \mapsto r y_1 \qquad \text{and} \qquad y_2 \mapsto r y_2$$

we read off from (2.23a) that

$$\rho_{++}^{T(4)}(r) = \frac{\zeta^4}{r^{2\Gamma-2}} \Bigg[\int_{\sigma/r}^{1-\sigma/r} dy_1 \int_{1+\sigma/r}^{\infty} dy_2 \\ \times \left(\frac{y_2 - y_1}{(y_2 - 1)\, y_2 (1 - y_1)\, y_1} \right)^{\Gamma} - \frac{r^{2\Gamma-2}}{(\Gamma-1)^2} \Bigg] \qquad \text{(A1)}$$

The fixed root points are now at 0 and 1. Let μ be a positive constant such that $\sigma/r < \mu \ll 1$. Then for the region of integration $\mu < y_1 < 1 - \mu$ and $y_2 > 1 + \mu$ in (A1) the integrand is bounded uniformly in r and is integrable. Hence the contribution to the double integral is $O(1)$, and from (2.19) the corresponding contribution to $\Delta^{(4)}$ is $O(\zeta^4/(\Gamma-2))$. To analyze the contribution from the remaining region of integration we decompose it to the subregions indicated by the particle configurations in Fig. 2.

Consider subregion A in which

$$\sigma/r \leqslant y_1 \leqslant \mu \qquad \text{and} \qquad \sigma/\tau \leqslant y_2 - 1 \leqslant \mu$$

Since y_1 and $y_2 - 1$ are small variables, we can expand the integrand:

$$\left(\frac{y_2 - y_1}{(y_2 - 1)\, y_2 (1 - y_1)\, y_1} \right)^{\Gamma} \sim \frac{1}{(y_2 - 1)^{\Gamma}\, y_1^{\Gamma}} [1 + \Gamma y_1 (y_2 - 1) + \ldots]$$

This shows that the leading large-r contribution to the double integral from subregion A is

$$\frac{r^{2\Gamma-2}}{(\Gamma-1)^2} + \frac{\Gamma}{(\Gamma-2)^2} (r^{\Gamma-2} - 1)^2 \qquad \text{(A2)}$$

Substituting (A1) in (A2) shows that the first term cancels. To leading order for large r the second term gives a contribution

$$\rho_{++}^{T(4)}(r) \sim \frac{\zeta^4 \Gamma}{(\Gamma-2)^2 r^2}$$

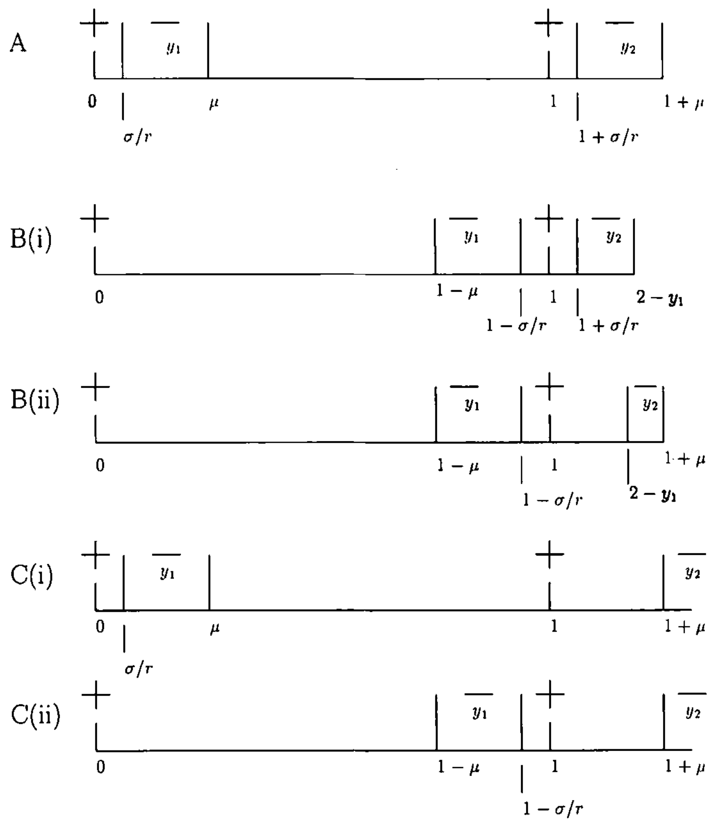

Fig. 2. Some regions of integration in the integral (A1) for $\rho^{T(4)}_{++}(r)$.

The first moment of this term (considered as a function defined on $[\sigma, \infty))$ is infinite. However, the entire second term in (A2) is exactly canceled in the formula (2.24) for $C(r)$ by an identical term contributing to $\rho^{(4)T}_{+-}(r)$, so there is no net contributed to $\Delta^{(4)}$.

Next consider the subregion B(i) in which

$$\sigma/r \leqslant 1 - y_1 < y \qquad \text{and} \qquad \sigma/r \leqslant y_2 - 1 \leqslant 1 - y_1 \tag{A3}$$

Expanding the integrand in (A1) given these conditions, we have

$$\left(\frac{y_2 - y_1}{(y_2 - 1)\, y_2 (1 - y_1)\, y_1}\right)^{\Gamma} \sim \frac{1}{(1 - y_1)^{\Gamma}} \left(1 + \Gamma \frac{1 - y_1}{y_2 - 1} + \dots\right)$$

The corresponding contribution to the double integral is

$$\int_{1-\mu}^{1-\sigma/r} \frac{dy_1}{(1-y_1)^{\Gamma}} \int_{1+\sigma/r}^{1+(1-y_1)} dy_2$$
$$+\Gamma \int_{1-\mu}^{1-\sigma/r} \frac{dy_1}{(1-y_1)^{\Gamma-1}} \int_{1+\sigma/r}^{1+(1-y_1)} \frac{dy_2}{y_2-1} \tag{A4}$$

When evaluated for large r these two integrals give

$$\frac{(r/\sigma)^{\Gamma-2}}{\Gamma-2} \tag{A5a}$$

and

$$\Gamma\left(-\frac{1}{(\Gamma-2)^2}\left[\left(\frac{r}{\sigma}\right)^{\Gamma-2}-1\right]+\frac{1}{\Gamma-2}\left(\frac{r}{\sigma}\right)^{\Gamma-2}\log r\right) \tag{A4b}$$

respectively. The corresponding contribution to $\Delta^{(4)}$ from (A5a) is $O(1/(\Gamma-2)^2)$, while the contribution from (A5b) is $O(1/(\Gamma-2)^3)$. Hence (A5a) does not contribute to $C_{\Delta}^{(4)}(r)$.

The analysis for region B(ii) proceeds analogously to that of region B(i) just presented. We again find contributions (A5a) and (A5b). In regions C(i) and C(ii) we find a contribution to $\Delta^{(4)}$ which is $O(1/(\Gamma-2)^2)$. This region does not contribute to $C_{\Delta}^{(4)}(r)$, and since all regions are now exhausted, the terms in the asymptotic expansion of $\rho_{++}^{T(4)}(r)$ which contribute to $\Delta^{(4)}$ give a contribution to the latter which is $O(1/(\Gamma-2)^3)$. Adding all such terms, including the term corresponding to (A2) which gives a divergent contribution to Δ,$^{(4)}$ we obtain the expansion

$$\rho_{++}^{T(4)}(r) \sim \zeta^4\left(\frac{\Gamma}{(\Gamma-2)^2}\frac{1}{r^{2\Gamma-2}}\left[1-\left(\frac{r}{\sigma}\right)^{\Gamma-2}\right]^2\right.$$
$$\left.+\frac{2\Gamma}{r^{2\Gamma-2}}\left\{-\frac{1}{(\Gamma-2)^2}\left[\left(\frac{r}{\sigma}\right)^{\Gamma-2}-1\right]+\frac{1}{\Gamma-2}\left(\frac{r}{\sigma}\right)^{\Gamma-2}\log r\right\}\right) \tag{A6a}$$

The first term originates from subregion A, while the remaining two terms have equal contributions from subregions B(i) and B(ii).

A similar approach suffices to analyze the terms in the asymptotic expansions of (2.23c) and (2.23b) which contribute to $\Delta^{(4)}$. We find

$$\rho_{-+}^{T(4)}(r) \sim \frac{\zeta^4\Gamma}{(\Gamma-2)^2\, r^{2\Gamma-2}}\left[\left(\frac{r}{\sigma}\right)^{\Gamma-2}-1\right]^2 \tag{A6b}$$

and

$$\rho^{T(4)}_{+-}(r) \sim \zeta^4 \left(\frac{\Gamma}{(\Gamma-2)^2} \frac{1}{r^{2\Gamma-2}} \left[1 - \left(\frac{r}{\sigma}\right)^{\Gamma-2} \right]^2 + \frac{8\Gamma}{r^{2\Gamma-2}} \left\{ - \frac{1}{(\Gamma-2)^2} \left[\left(\frac{r}{\sigma}\right)^{\Gamma-2} - 1 \right] + \frac{1}{\Gamma-2} \left(\frac{r}{\sigma}\right)^{\Gamma-2} \log r \right\} \right) \tag{A6c}$$

The particle configurations corresponding to the subregions of integration contributing to the terms in these expansions are given in Fig. 3.

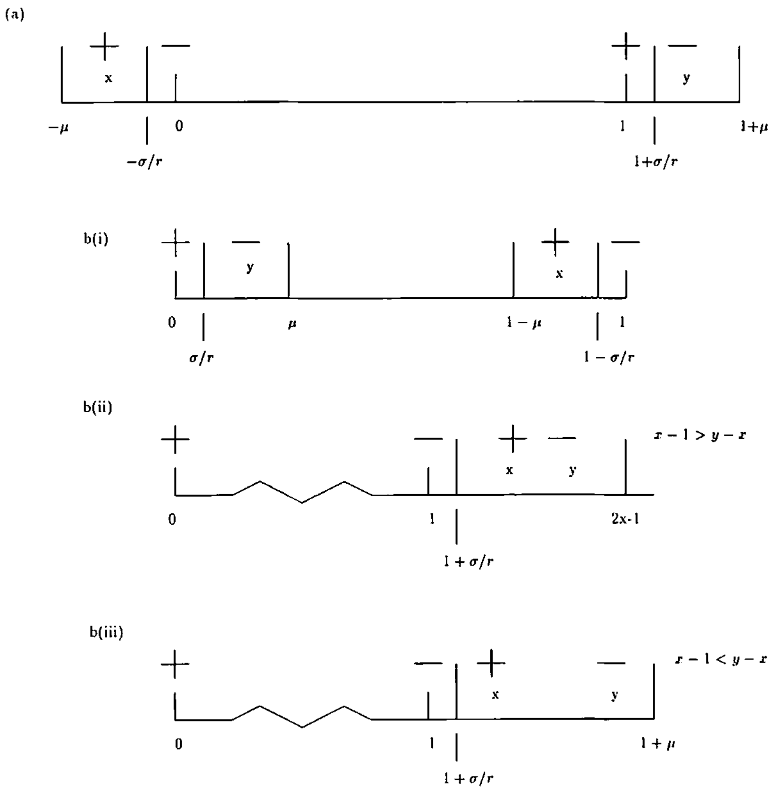

Fig. 3. The configuration (a) gives the term in (A6b), while the configuration b(i) gives the first term in (A6c). The configurations b(ii) and b(iii) both contribute equally to the remaining terms in (A6c). These configurations each have a multiplicity factor of 4, which corresponds to the four ways of placing the mobile pair x and y on either side of the two root charges.

Forming the combination (2.24) of (A6a), (A6b), and (A6c) required to form $C(r)$, we see that a lot of cancellation takes place, and the result (2.27) of the text results. Furthermore, it is important to observe that due to the cancellations only $\rho^{T(4)}_{+-}(r)$ contributes to $C^{(4)}_{\Delta}$ and this contribution is restricted to the region of integration depicted in b(ii) of Fig. 3. Analogous to (A5b), this contribution can be written as a double integral, which is given by (2.28) of the text. The integrand of the double integral is obtained from the expansion of the integrand (excluding the factor of $1/r^{\Gamma}$) in the second double integral of (2.23b):

$$\left(\frac{x(y-r)}{(y-x)\,y(x-r)}\right)^{\Gamma}-\left(\frac{1}{y-x}\right)^{\Gamma}\sim\frac{\Gamma}{(x-r)(y-x)^{\Gamma-1}} \tag{A7}$$

valid for $x-r$ and $y-x$ small and positive with $y-x<x-r$. The domain of integration is that implied in configuration b(ii) of Fig. 3.

APPENDIX B

Our objective here is to study the leading large-r behavior of the formula (3.4) for $C^{(4)}(r)$ as a function of Γ. We begin by changing variables $x_1 \mapsto ru$, $x_2 \mapsto rv$, so that (3.4) reads

$$C^{(4)}(r)=\frac{q^2\zeta^4}{r^{2\Gamma-2}}\int^{*} du\,dv\left(\left|\frac{u-v}{(v-1)(u-1)\,uv}\right|^{\Gamma}\right.$$
$$\left.-2S\left|\frac{u(v-1)}{(u-v)(u-1)v}\right|^{\Gamma}-\frac{2}{|u-v|^{\Gamma}}\right) \tag{B1}$$

where * denotes the integration region

$$\{(u,v)\in \mathrm{R}^2\colon |u-v|>\sigma/r,\ |u|,\ |v|>\sigma/r,\ |u-1|,\ |v-1|>\sigma/r\}$$

Due to the integrand being symmetric in u and v, and unchanged by the mapping $u\mapsto 1-u$, $v\mapsto 1-v$, we can restrict the integration region to

$$\{(u,v)\in \mathrm{R}^2\colon u>v,\ v>1-u,\ |u-v|>\sigma/r,\ |u-1|,\ |v-1|>\sigma/r\} \tag{B2}$$

provided we multiply the integral by 4. Let us divide the integration region (B2) into labeled and unlabeled regions as indicated in Fig. 4.

All regions outside those labeled in Fig. 4 give a contribution to $C^{(4)}(r)$ which is $O(1/r^{2\Gamma-2})$. This follows from (B1), since the range of the integrand is independent of r in these regions. The range of the integrand is not independent of r in the labeled regions of Fig. 4, and consequently further calculations are needed.

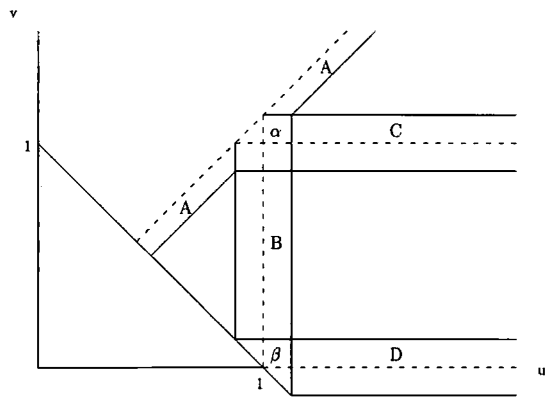

Fig. 4. Regions of integration in (B2) which are used to analyze the integral (B1) for $C^{(4)}(r)$. A distance σ/r either side of each dashed line bordering or contained within a region is to be excluded.

Let us illustrate our method of analyzing the leading-order contribution from the labeled regions by considering region D. The total contribution to $C^{(4)}(r)$ from this region is given by

$$\frac{q^2\zeta^4}{r^{2\Gamma-2}}\int_{1+\mu}^{\infty} du\left(\int_{\sigma/r}^{\mu}+\int_{-\mu}^{-\sigma/r}\right) dv\, F(u, v) \tag{B3}$$

where $F(u, v)$ is the integrand in (B1). For large r the leading contribution to (B3) comes from the neighborhood of $|v| = \sigma/r$. We can thus ignore the last term in (B1), and the term implicit in the symmetrized form of the second term in (B1). Expanding the remaining two terms for small v gives

$$\frac{2\Gamma}{|v|^{\Gamma}(u-1)^{\Gamma}}\left[-\frac{v}{u}+v-\frac{1}{2}\left(\frac{v}{u}\right)^2+\frac{v^2}{2}+O(v^3)\right]$$

in place of $F(u, v)$ in (B3).

Evaluating this integral to leading order for large r gives behaviors

$$\begin{aligned} &O\left(\frac{1}{r^{2\Gamma-2}}\right), \quad 2<\Gamma<3 \\ &O\left(\frac{\log r}{r^4}\right), \quad \Gamma=3 \\ &O\left(\frac{1}{(\Gamma-3)\, r^{\Gamma+1}}\right), \quad \Gamma>3 \end{aligned} \tag{B4}$$

Proceeding similarly, we find the leading large-r contributions from the labeled regions of Fig. 4 to be as in Table I.

Table I

Region	Contribution to $C^{(4)}(r)$		
	$2<\Gamma<3$	$\Gamma=3$	$\Gamma>3$
A, B, C, D	$O\left(\frac{1}{r^{2\Gamma-2}}\right)$	$O\left(\frac{\log r}{r^4}\right)$	$O\left(\frac{1}{(\Gamma-3)\, r^{\Gamma+1}}\right)$
α	$O\left(\frac{1}{r^{\Gamma}}\right)$	$O\left(\frac{1}{r^3}\right)$	$O\left(\frac{1}{r^{\Gamma}}\right)$
β	$O\left(\frac{1}{r^{2\Gamma-2}}\right)$	$O\left(\frac{\log^2 r}{r^4}\right)$	$O\left(\frac{1}{(\Gamma-3)^2\, r^4}\right)$

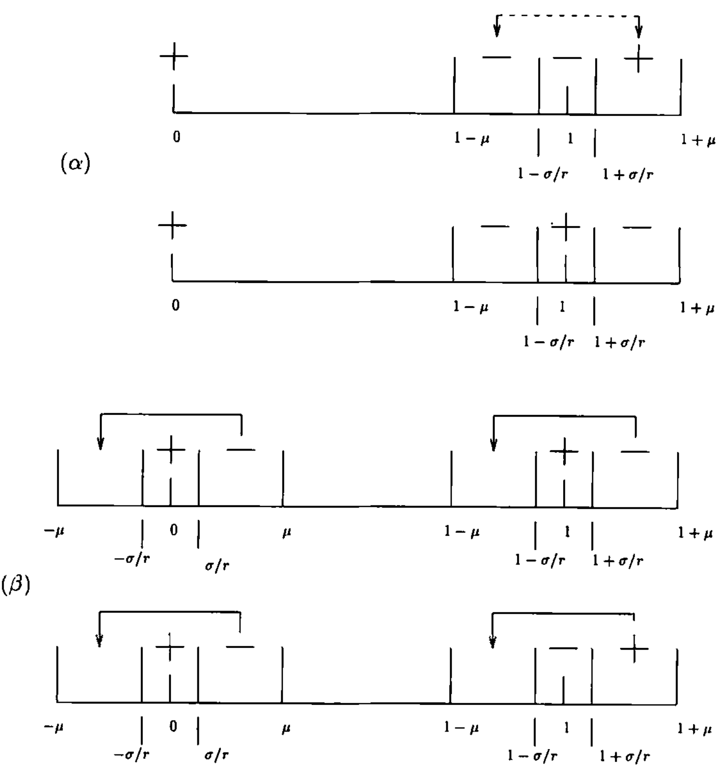

Fig. 5. Charge configurations corresponding to regions α and β in Fig. 4, which give the leading-order contribution to $C^{(4)}(r)$ for $2<\Gamma<4$ and $\Gamma>4$, respectively. The arrows indicate alternative placement of mobile charges.

Hence the leading-order contribution to $C^{(4)}(r)$ comes from region α for $2<\Gamma<4$ and from region β for $\Gamma>4$. These regions correspond to the charge configurations of Fig. 5.

ACKNOWLEDGMENTS

We thank the referees for their considered comments. P. J. F. was supported by the Australian Research Council.

REFERENCES

1. P. W. Anderson, G. Yuval, and D. R. Hamman, *Phys. Rev. B* **1**:4464 (1970).
2. A. Schmid, *Phys. Rev. Lett.* **51**:1506 (1983).
3. H. Schultz, *J. Phys. A* **14**:3277 (1981).
4. J. M. Kosterlitz, *J. Phys. C* **7**:1046 (1974).
5. P. Minnhagen, *Rev. Mod. Phys.* **59**:1001 (1987).
6. K. D. Schotte and U. Schotte, *Phys. Rev. B* **4**:2228 (1971).
7. A. Alastuey and F. Cornu, *J. Stat. Phys.* **66**:165 (1992).
8. P. J. Forrester, *J. Stat. Phys.* **42**:871 (1986).
9. P. J. Forrester, *J. Stat. Phys.* **45**:153 (1986).
10. P. J. Forrester, *J. Stat. Phys.* **59**:57 (1989).
11. M. Gaudin, *J. Phys.* (Paris) **46**:1027 (1985).
12. E. Meeron, *J. Chem. Phys.* **28**:630 (1958); R. Abe, *Prog. Theor. Phys.* **22**:213 (1959).
13. A. Alastuey and Ph. A. Martin, *Phys. Rev. A* **40**:6485 (1989).
14. E. R. Speer, *J. Stat. Phys.* **42**:895 (1986).
15. P. J. Forrester and B. Jancovici, *J. Stat. Phys.* **69**:163 (1992).
16. J. M. Kosterlitz and D. J. Thouless, *J. Phys. C* **6**:1181 (1973).
17. Ph. A. Martin, *Rev. Mod. Phys.* **60**:1075 (1988).
18. B. Jancovici, *J. Stat. Phys.* **80** (1995).
19. A. Alastuey, F. Cornu, and A. Perez, *Phys. Rev. E* **49**:1077 (1994).
20. P. J. Forrester, *J. Stat. Phys.* **51**:457 (1988).
21. F. Cornu and B. Jancovici, *J. Chem. Phys.* **90**:2444 (1989).
22. M. L. Rosinberg, Unpublished.
23. G. Manificat and J. M. Caillol, *J. Stat. Phys.* **74**:1309 (1994).
24. A. Alastuey, F. Cornu, and B. Jancovici, *Phys. Rev. A* **38**:4916 (1988).
25. J. M. Caillol, *J. Chem. Phys.*, to appear.
26. J. M. Caillol and D. Levesque, *Phys. Rev. B* **33**:499 (1986).
27. Y. Levin, X. Li, and M. E. Fisher, *Phys. Rev. Lett.* **73**:2716 (1994).
28. R. Evans and L. de Carvalho, *Mol. Phys.*, to appear.